Rough-pipe flows and the existence of fully developed turbulence

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It is widely believed that at high Reynolds number (Re) all turbulent flows approach a limiting state of "fully developed turbulence" in which the statistics of the velocity fluctuations are independent of Re. Nevertheless, direct measurements of the velocity fluctuations have failed to yield firm empirical evidence that even the second-order structure function becomes independent of Re at high Re, let alone structure functions of higher order. Here we relate the friction coefficient (f) of rough-pipe flows to the second-order structure function. Then we show that in light of experimental measurements of f our results yield unequivocal evidence that the second-order structure function becomes independent of Re at high Re, compatible with the existence of fully developed turbulence. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2189285]

A conspicuous manifestation of turbulence, and one that lends itself readily to theoretical analysis, is the advent of fluctuations in the velocity field of a flow. In a classic paper,[1] Kolmogórov made a few plausible assumptions to show that in flows of high Re the statistics of the velocity fluctuations might become asymptotically invariant to further increases in Re. Kolmogórov's Re-independent statistics of the velocity fluctuations define a limiting state of "fully developed turbulence" that is widely believed to exist at very high Re. Support for the existence of fully developed turbulence has been sought in direct, hot-wire measurements of the velocity fluctuations. These efforts have been concerned mostly with the second-order structure function, which is just one component of the statistics of the velocity fluctuations; yet, as recent research has shown,[2–5] the results remain inconclusive. In this Brief Communication, we concentrate on the second-order structure function and seek to prove that it does become independent of Re at high Re. To that end, we formulate a theory that allows us to harness empirical results other than direct measurements of the velocity fluctuations. We start with an outline of the intricacies of the problem.

Kolmogórov studied the statistics of the velocity fluctuations at the length scale l, u_l. He used dimensional analysis to show that the second-order structure function $\overline{(u_l)^2}$ [i.e., the mean value of $(u_l)^2$] must take the form[1]

$$\overline{(u_l)^2} = P[\mathrm{Re}, l/L](\varepsilon l)^{2/3}, \qquad (1)$$

where P is a dimensionless function of the dimensionless variables Re and l/L, $\mathrm{Re} \gg 1$ is a Reynolds number of the flow, L is the largest length scale in the flow, ε is the mean value of the rate of energy dissipation per unit mass, and l is confined to the inertial range, $L \gg l \gg \eta$, where η is the viscous (dissipation) length scale.[6] Given that $\mathrm{Re} \gg 1$ and $l/L \ll 1$, it is natural to identify a plausible asymptotic scenario for $\mathrm{Re} \to \infty$ and $l/L \to 0$. To that end, Kolmogórov assumed complete similarity with respect to Re and l/L, or $\lim_{\mathrm{Re}\to\infty} \lim_{l/L\to 0} P = p$, where $p > 0$ is a constant prefactor.[7] Under this assumption, the *leading term* in the asymptotic expansion of $\overline{(u_l)^2}$ is $\overline{(u_l)^2}_{\mathrm{lt}} = p(\varepsilon l)^{2/3}$, *independent of* Re, and therefore compatible with the existence of fully developed turbulence.[8]

Shortly after the publication of Kolmogórov's paper in 1941, it was objected that the asymptotic scenario proposed by Kolmogórov, and customarily known as K41, could not account for intermittency[9]—a phenomenon whereby the rate of energy dissipation per unit mass fluctuates around its mean value, ε. To account for intermittency, Kolmogórov himself argued that the similarity with respect to l/L might be incomplete, and went on to assume the simplest type of incomplete similarity[7] with respect to l/L. Under this assumption, the leading term of $\overline{(u_l)^2}$ is $\overline{(u_l)^2}_{\mathrm{lt}} = p(\varepsilon l)^{2/3}(l/L)^{\alpha}$, where α is a constant intermittency exponent.[10] This alternative asymptotic scenario, known as K62, has led to a vast body of research on intermittency.[9] Nevertheless, in K62 $\overline{(u_l)^2}_{\mathrm{lt}}$ remains independent of Re, as was the case in K41, and therefore continues to be compatible with the existence of fully developed turbulence (albeit not the same fully developed turbulence predicted by K41). In this sense, K62 represents only a minor departure from K41.

More recently, a major departure from K41 has been suggested on the basis of new experimental results.[2,3] These results indicate that even at very high Re the prefactor p is not constant, but subject to a discernible dependence on Re—a dependence that is marked enough as to cast doubts on the existence of fully developed turbulence. To analyze these results, Barenblatt and Goldenfeld[4] argued that there might be no similarity with respect to Re. Further, they argued that the form of $\overline{(u_l)^2}$ at high Re should be invariant under a natural redefinition of Re,[11] and showed that for this *principle of asymptotic covariance* to hold $\overline{(u_l)^2}$ (and therefore p) must depend on Re only through ln Re. Last, they wrote[12] $p(\ln \mathrm{Re}) = p_0 + p_1\delta + o(\delta)$, where $\delta \equiv 1/|\ln \mathrm{Re}| \ll 1$. Under these conditions,

$$\overline{(u_l)^2} = \left(p_0 + \frac{p_1}{\ln \mathrm{Re}}\right)(\varepsilon l)^{2/3}(l/L)^{\alpha} + \mathrm{o}\left(\frac{1}{\ln \mathrm{Re}}\right), \quad (2)$$

and the behavior of $\overline{(u_l)^2}$ at high Re depends crucially on the value of $p_0 \geqslant 0$. If $p_0 > 0$, then $\overline{(u_l)^2}_{\mathrm{lt}} = p_0(\varepsilon l)^{2/3}(l/L)^{\alpha}$, independent of Re, and is therefore compatible with the existence of the same fully developed turbulence predicted by K41 (if $\alpha=0$) or K62 (if $\alpha \neq 0$). On the other hand, if $p_0=0$, then $\overline{(u_l)^2}_{\mathrm{lt}} = p_1(\varepsilon l)^{2/3}(l/L)^{\alpha}/\ln \mathrm{Re}$, dependent on Re, and is therefore incompatible with the existence of fully developed turbulence. To decide between these alternative scenarios, Barenblatt and Goldenfeld[4] computed best fits of (2) to hot-wire data from a large wind tunnel and the atmosphere,[2] for both $p_0 > 0$ and $p_0=0$, and concluded that the data were "not inconsistent with either of the two possibilities." In another attempt at deciding the matter, Sreenivasan[5] studied a large set of hot-wire data, and concluded that the prefactor p is "*more or less* universal, *essentially* independent of the flow as well as the Reynolds number".[13] Nevertheless, he noted that the scatter in the data was large,[14] and that to evince the behavior of p for $\mathrm{Re} \gg 1$ one would have "to cover a wide range of Reynolds numbers in a single, well-controlled flow, and use instrumentation whose resolving power and quality remains equally good in the entire range"; unfortunately, "such experiments and efforts are not yet in the horizon at present."[5]

To decide the matter, we intend to resort to experimental data on the friction coefficient of rough pipes, f. These data[15,16] appear to be well suited to our purpose: they contain very little scatter and show beyond doubt that, for any fixed wall roughness, the leading term of f is independent of Re at high Re. Further, we know that for the case $p_0 > 0$ (and $p_1 = \alpha = 0$) the observed behavior of f at high Re can be predicted theoretically by establishing a relation[17] between f and $\overline{(u_l)^2}$. Our problem is to make a similar prediction for the case $p_0=0$; our hope is that this prediction will turn out to be at odds with the observed behavior of f at high Re. Such an outcome would allow us to rule out the case $p_0=0$, and therefore to prove that $\overline{(u_l)^2}$ becomes independent of Re at high Re.

The friction coefficient of a pipe may be defined as $f \equiv \tau/\rho V^2$, where τ is the shear stress on the wall of the pipe, ρ is the density of the liquid flowing through the pipe, and V is the average velocity of the flow. We seek to obtain an expression relating f to the second-order structure function of (2). Now (2) was originally derived under the assumptions of isotropy and homogeneity, but the turbulent flow in a pipe is both anisotropic and inhomogeneous. Nevertheless, recent research[18] has established that (2) applies as well to flows that are neither isotropic nor homogeneous.[19] Further, if v_l denotes the characteristic velocity of a turbulent eddy of size l, we may identify[9] $v_l = \sqrt{\overline{(u_l)^2}}$, where $\overline{(u_l)^2}$ is given by (2) with $L=D$ (the diameter of the pipe, which sets the largest length scale in the flow). Therefore, it follows from (2) that, regardless of the value of p_0, the smaller the eddy the lower its velocity. With these considerations in mind, we now seek to derive an expression for τ, the shear stress on the rough wall of the pipe.

Let us call S the wetted surface tangent to the peaks of the roughness elements of the wall, Fig. 1. (We assume roughness elements of uniform size r, as in Nikuradse's experiments.[15]) Then, for $\mathrm{Re} \gg 1$, the shear stress is effected by momentum transfer across S. Above S, the velocity of the flow scales with V, and the fluid carries a high horizontal momentum per unit volume ($\sim \rho V$). Below S, the velocity of the flow is negligible, and the fluid carries a negligible horizontal momentum per unit volume. Now consider an eddy that straddles the wetted surface S (with one half of the eddy above S, and the other half below). This eddy transfers fluid of high horizontal momentum downwards across S, and fluid of negligible horizontal momentum upwards across S. The net rate of momentum transfer across S is set by the velocity normal to S, which velocity is provided by the eddies that straddle S. Therefore, if v_n denotes the velocity normal to S provided by the *dominant* eddy that straddles S, then the shear stress effected by momentum transfer across S scales in the form $\tau \sim \rho V v_n$. Now the size of the largest eddy that straddles S scales with r, the size of the roughness elements. This eddy provides a velocity $v_r = \sqrt{\overline{(u_r)^2}}$ normal to S, where $\overline{(u_r)^2}$ is given by (2) with $l=r$ and $L=D$. Smaller eddies do provide a velocity normal to S, but these velocities are overwhelmed by the velocity of the eddy of size r. (Recall that the smaller the eddy the lower its velocity.) Thus $v_n \sim v_r$, and the dominant eddy that straddles S is the largest eddy that straddles S. We conclude[17,20] that $\tau \sim \rho v_r V$, and therefore $f \sim v_r/V = \sqrt{\overline{(u_r)^2}}/V$.

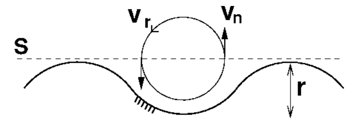

FIG. 1. Immediate vicinity of a rough-pipe wall with roughness elements of uniform size r. The dashed line is the trace of the wetted surface S.

To complete our derivation, we relate ε to V and D using the phenomenological theory, which is based on two tenets pertaining to the steady production of turbulent (kinetic) energy: (1) the production occurs at the length scale of the largest eddies in the flow and (2) the rate of production is independent of the viscosity. From these tenets and the equality of production and dissipation, it follows that we can obtain a scaling expression for ε, the rate of dissipation of turbulent energy per unit mass of liquid, in terms of the velocity of the largest eddies (which $\sim V$) and of the size of the largest eddies (which $\sim D$).[21] The largest eddies possess a kinetic energy per unit mass $e \sim V^2$ and a turnover time $t \sim D/V$. These eddies persist for a time t, whereupon they split into eddies of size $\sim D/2$, thereby transferring their energy to smaller length scales. For the steady state to be preserved, a new set of large eddies must be produced at time intervals t, implying that[21] $\varepsilon = e/t \sim V^3/D$. By using $\varepsilon \sim V^3/D$, $l=r$, and $L=D$ in (2), and substituting the result in $f \sim \sqrt{\overline{(u_r)^2}}/V$, we obtain

$$f \sim (r/D)^{1/3+\alpha/2}\left(p_0 + \frac{p_1}{\ln \mathrm{Re}}\right)^{1/2}, \quad (3)$$

an expression relating f to the parameters p_0, p_1, and α of the asymptotic expansion of the structure function.

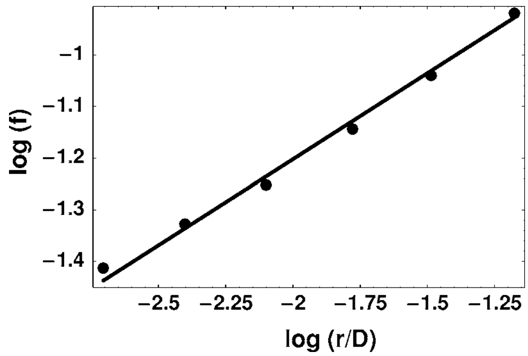

FIG. 2. A comparison of Nikuradse's data (Ref. 15) with Strickler's expression. The straight line is $\log(f) = -0.54 + \log(r/D)/3$. The data points correspond to the highest Re ($\approx 10^6$) tested by Nikuradse.

Now consider a pipe of fixed roughness, $r/D = \text{const} \ll 1$. If $p_0 = 0$, the leading term of f depends on Re: $f_{\text{lt}} \sim \sqrt{p_1}(r/D)^{1/3+\alpha/2}(\ln \text{Re})^{-1/2}$. Thus for $p_0 = 0$ the friction coefficient vanishes asymptotically at high Re, a conclusion that is at odds with all experimental data on rough-pipe flows. On the other hand, if $p_0 > 0$ the leading term of f is independent of Re: $f_{\text{lt}} \sim \sqrt{p_0}(r/D)^{1/3+\alpha/2}$. Thus for $p_0 > 0$ the friction coefficient tends to a positive constant at high Re, a conclusion that is qualitatively consistent with all experimental data on rough-pipe flows. Further, in the case $\alpha = 0$, $f_{\text{lt}} \sim \sqrt{p_0}(r/D)^{1/3}$, which we recognize as Strickler's empirical expression[22,17,15] for the friction coefficient of a pipe of roughness r/D at high Re. In the case $\alpha \neq 0$, $f_{\text{lt}} \sim \sqrt{p_0}(r/D)^{1/3+\alpha/2}$, which is a generalized form of Strickler's empirical expression that accounts for the effect of intermittency. Given that the experimental data can be fitted very well even if α is set to zero (Fig. 2), we infer that $1/3 \gg |\alpha|/2$, or $|\alpha| \ll 2/3$, consistent with the available estimates of α.[12]

From the previous paragraph, we conclude that Eq. (3) together with the experimental data embodied by Strickler's empirical expression allows us to establish that $p_0 > 0$. Therefore, the leading term of the structure function $\overline{(u_l)^2}$ is independent of Re, compatible with the existence of fully developed turbulence.

The logic of our reasoning so far has been the following. We have shown theoretically that $p_0 > 0$ is a *necessary* condition for the friction coefficient of a rough pipe to tend to a positive constant at high Re. Then, given the unequivocal experimental evidence that in a rough pipe $\lim_{\text{Re}\to\infty} f = \text{const} > 0$, we have concluded that it must be that $p_0 > 0$. Interestingly, $p_0 > 0$ is not in general a *sufficient* condition for the friction coefficient of a pipe to tend to a positive constant at high Re: the pipe must be rough. To elucidate this statement, we recall our scaling expression for the shear stress, $\tau \sim \rho v_r V$. This expression indicates that the momentum transfer is dominated by the eddies of size r-the same size as the roughness elements. Now in a rough-pipe flow of sufficiently high Re, r exceeds the viscous length scale, $r > \eta$. In fact, from $\eta \equiv \nu^{3/4}\varepsilon^{-1/4}$ (where ν is the kinematic viscosity) and $\varepsilon \sim V^3/D$, we can write $\eta/D \sim (\nu/VD)^{3/4} = \text{Re}^{-3/4}$ and conclude that for any given r the condition $r > \eta$ holds at sufficiently high Re. If Re is increased further, η lessens and newer, smaller eddies populate the flow. Yet the momentum transfer continues to be dominated by eddies of size r, and f remains invariant. This argument explains the behavior of f for all rough pipes, no matter how small the roughness elements. If $r = 0$, however, the condition $r > \eta$ cannot be attained, even at extremely high Re. Thus in a smooth pipe[23] the momentum transfer will always be dominated by the smaller eddies in the inertial range, eddies whose size scales with η; since η lessens as Re increases, the momentum transfer will be dominated by ever smaller eddies as Re increases, and $\lim_{\text{Re}\to\infty} f = 0$—even though $p_0 > 0$. To verify this conclusion mathematically, we study (3) for $r \sim \eta$. By substituting $\eta/D \sim \text{Re}^{-3/4}$ in place of r/D, we obtain

$$f \sim \text{Re}^{-1/4-3\alpha/8}\left(p_0 + \frac{p_1}{\ln \text{Re}}\right)^{1/2}. \qquad (4)$$

In accord with our discussion above, (4) indicates that in a smooth pipe the friction coefficient vanishes asymptotically at high Re, whether $p_0 = 0$ or $p_0 > 0$. Had we to decide between $p_0 = 0$ and $p_0 > 0$ on the basis of (4) and experimental data, the answer would not be clearcut, because $(\ln \text{Re})^{-1/2}$ varies but very slowly at high Re. Nevertheless, we have established previously that $p_0 > 0$. It follows that $f_{\text{lt}} \sim \sqrt{p_0}\text{Re}^{-1/4-3\alpha/8}$, which in the case $\alpha = 0$ coincides with Blasius's empirical expression[15,17] for the friction coefficient of a smooth pipe at high Re. In the case $\alpha \neq 0$, $f_{\text{lt}} \sim \sqrt{p_0}\text{Re}^{-1/4-3\alpha/8}$, which is a generalized form of Blasius's empirical expression that accounts for the effect of intermittency.

We have concerned ourselves with two aspects of turbulent flows: (1) small-scale statistics that might become independent of Re at high Re and (2) global properties that do become independent of Re at high Re. The former were first surmised in the early history of turbulence physics; they include the second-order structure functions associated with K41 and K62, the asymptotic scenarios of Kolmogórov. The latter have long been known to engineers; they include the drag of bluff bodies and the friction coefficient of rough conduits, among others.[24] Here we have established a relation between the two. In particular, we have found that for the friction coefficient of rough pipes to tend to positive constants at high Re, as seen in experiments, it must be that the second-order structure function becomes independent of Re at high Re, compatible with the existence of fully developed turbulence. In addition, we have found that the existence of fully developed turbulence is compatible with the experimental evidence that the friction coefficient of smooth pipes tends to zero at high Re.

Our findings support the existence of fully developed turbulence, but do not prove it. To prove the existence of fully developed turbulence, we must ascertain the behavior of the structure functions of order higher than 2. A promising way of ascertaining this behavior might involve the harnessing of empirical evidence other than direct measurements of the velocity fluctuations.

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[6]A similar expression holds for the structure function of order n, $\overline{(u_l)^n} = P_n[\mathrm{Re}, l/L](\varepsilon l)^{n/3}$.

[7]G. I. Barenblatt, *Scaling, Self-similarity, and Intermediate Asymptotics* (Cambridge University Press, Cambridge, UK, 1986), Chap. 10.

[8]"The mathematical expression of the assumption of a limiting state of fully developed turbulence is that statistical averages of the flow exhibit complete similarity with respect to Re" (Ref. 4). Therefore, in fully developed turbulence the leading term of $\overline{(u_l)^n}$ is independent of Re for all $n \geqslant 2$. Here we concern ourselves only with $n=2$.

[9]U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, UK, 1995).

[10]A. N. Kolmogórov, "A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number," J. Fluid Mech. **13**, 82 (1962).

[11]For example, the form of the structure function should not change if we used Re_λ in lieu of a global Re.

[12]Barenblatt and Goldenfeld also assumed a general type of incomplete similarity with respect to l/L in which the intermittency exponent, α, may depend on Re (Ref. 7). Then, in keeping with the principle of asymptotic covariance, they wrote $\alpha(\ln \mathrm{Re}) = \alpha_0 + \alpha_1/\ln \mathrm{Re} + o(1/\ln \mathrm{Re})$, and pointed out that the experimental data of Praskovsky and Oncley (Ref. 2) favor $\alpha_0 = 0$ (or $\alpha \to 0$ at high Re). Since α is of slight concern here, we continue to represent it as a constant. Nevertheless, in all our equations α may be substituted by $\alpha_0 + \alpha_1/\ln \mathrm{Re} + o(1/\ln \mathrm{Re})$.

[13]Italics in the original.

[14]The scatter may stem from simplifications variously used in processing hot-wire data; see S. T. Thoroddsen, "Reevaluation of the experimental support for the Kolmogorov refined similarity hypothesis," Phys. Fluids **7**, 691 (1995).

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[19]Our conclusions would still hold if we adopted one of several expressions that are similar to (2) but do not satisfy the principle of asymptotic covariance—e.g., $p(\mathrm{Re}) = p_0 + p_1 \mathrm{Re}^{-\beta}$, an expression that can be derived without assuming isotropy or homogeneity; see T. S. Lundgren, "Kolmogorov turbulence by the method of matched asymptotic expansions," Phys. Fluids **15**, 1074 (2003). For experiments that support the principle of asymptotic covariance see A. Arneodo, S. Manneville, J. F. Muzy, and S. G. Roux, "Revealing a lognormal cascading process in turbulent velocity statistics with wavelet analysis," Philos. Trans. R. Soc. London, Ser. A **357**, 2415 (1999).

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[22]Reprinted in A. Strickler, *Contribution to the question of a velocity formula and roughness data for streams, channels and close pipelines*, translation by T. Roesgen and W. R. Brownlie (Caltech, Pasadena, 1981).

[23]By "smooth" we mean "mathematically smooth." Whether an actual pipe may be considered smooth appears to be a question that grants no easy answers; see G. I. Barenblatt and A. J. Chorin, "Scaling of the intermediate region in wall-bounded turbulence: The power law," Phys. Fluids **10**, 1043 (1998); A. J. Smits and M. V. Zagarola, "Response to "Scaling of the intermediate region in wall-bounded turbulence: The power law," Phys. Fluids **10**, 1045 (1998); B. J. McKeon, M. V. Zagarola, A. J. Smits, "A new friction factor relationship for fully developed pipe flow," J. Fluid Mech. **538**, 429 (2005).

[24]For example, the normalized dissipation in the experiments of D. P. Lathrop, J. Fineberg, and H. L. Swinney, "Turbulent flow between concentric rotating cylinders at large Reynolds number," Phys. Rev. Lett. **68**, 1515 (1992) and O. Cadot, Y. Couder, A. Daerr, S. Douady, and A. Tsinober, "Energy injection in closed turbulent flows: Stirring through boundary layers versus inertial stirring," Phys. Rev. E **56**, 427 (1997).