SPECIAL RELATIVITY HOMEWORK – WEEK 5

Exercise 1. *In this exercise we consider an off-shell electromagnetic potential $A_\mu(x)$, with 4 independent components per spacetime point, unconstrained by any field equations.*

1. *The field strength $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ does satisfy a differential equation, namely:*

$$\partial_{[\mu}F_{\nu\rho]} = 0 \ . \tag{1}$$

   Make sure you understand why. How many components does eq. (1) have? Write these components in non-relativistic notation, in terms of the electric and magnetic fields $\mathbf{E}, \mathbf{B}$.

2. *Count the components of a general antisymmetric matrix field $F_{\mu\nu}(x)$, and subtract from these the number of constraints in (1). Compare to the 4 components of $A_\mu(x)$. What are the reasons for the discrepancy?*

Exercise 2. *The most general action of electromagnetism coupled to charged matter can be written as:*

$$S = -\frac{1}{4}\int d^4x \, F_{\mu\nu}F^{\mu\nu} + S_{interaction} \ . \tag{2}$$

We can define the electromagnetic 4-current $J^\mu(x)$ as the variational derivative of $S_{interaction}$ w.r.t. A_μ:

$$\delta S_{interaction} = \int d^4x \, J^\mu(x)\delta A_\mu(x) \ . \tag{3}$$

1. *Derive the field equations that arise from varying the full action (2) w.r.t. A_μ. Write their components in non-relativistic notation, in terms of $\mathbf{E}$, $\mathbf{B}$, and the components $(\rho, \mathbf{j})$ of J^μ.*

2. *Show that these field equations imply a conservation law $\partial_\mu J^\mu = 0$. Rewrite this conservation law in terms of the components $(\rho, \mathbf{j})$.*

3. *Derive the same conservation law from the demand that $S_{interaction}$ is invariant under infinitesimal gauge transformations $\delta A_\mu(x) = -\partial_\mu \epsilon(x)$.*

4. *Consider a charged particle with $S_{interaction} = q\int A_\mu dx^\mu$. Let $\mathbf{x} = \mathbf{x}_{particle}(t)$ be the particle's trajectory. Derive $J^\mu = (\rho, \mathbf{j})$ from the definition (3), and show that it agrees with the undergrad notions of charge density and current density.*