

SPECIAL RELATIVITY HOMEWORK – WEEK 3

In the lecture, we inadvertently derived the formula for relativistic energy-momentum:

$$p^\mu = (E, \mathbf{p}) = \frac{m}{\sqrt{1 - \mathbf{v}^2}}(1, \mathbf{v}) . \quad (1)$$

Exercise 1. Consider a gas of n particles per unit volume. Each particle has the same mass m and velocity of magnitude v . The directions of the particles' velocities are uniformly distributed. Show that the stress-energy tensor has the form:

$$T^{\mu\nu} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} , \quad (2)$$

and find the values of ε and P . What is the physical meaning of these components?

Exercise 2. Now, take the limits $m \rightarrow 0$ and $v \rightarrow 1$, such that the energy E and momentum $|\mathbf{p}|$ of each particle remain finite. What happens to the length of the 4-momentum $p^\mu = (E, \mathbf{p})$? What happens to the trace T^μ_μ of the gas' stress-energy tensor? Do you understand how the two facts are related?

Exercise 3. In this exercise, we expand on our discussion of non-orthonormal bases in Euclidean space. As introduced in the lecture, a set of basis vectors $\mathbf{e}_i$ defines a dual basis $\mathbf{e}^i$, via:

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j . \quad (3)$$

An arbitrary vector $\mathbf{v}$ can be expressed in components w.r.t. each of the bases, as:

$$\mathbf{v} = v^i \mathbf{e}_i = v_i \mathbf{e}^i . \quad (4)$$

Now, let us define the pair of matrices:

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j ; \quad g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j . \quad (5)$$

From these definitions, derive the following (closely related) statements:

1. Index raising/lowering works as $v_i = g_{ij} v^j$ and $v^i = g^{ij} v_j$.
2. g^{ij} is the matrix inverse of g_{ij} .
3. g_{ij} , δ_i^j and g^{ij} are all related to each other by index raising/lowering.