

Nichols algebras and Kazhdan-Lusztig correspondences

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Representation Theory Seminar
OIST, 12.10.2026

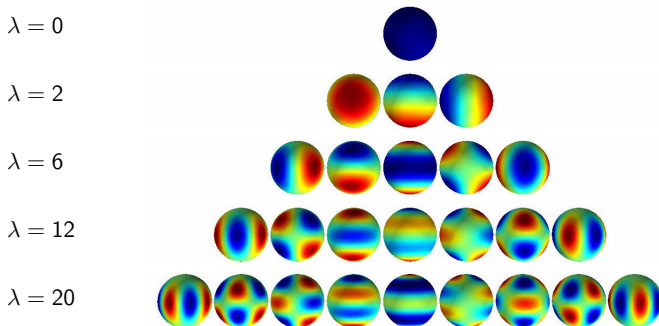
Motivation

"Algebra and representation theory is a great tool to analyze the phenomenon of symmetry."



Motivation

"If a differential equation has a group of symmetries then its spaces of solutions become representations"



Eigenfunctions of the spherical Laplace operator $\Delta f = -\lambda f$.
Eigenspaces carry an action of the group of rotations SO_3 .

The representation theory predicts dimensions, eigenvalues, product rules.

We now specify for this talk:

What kind of representation theory?

What kind of analysis or physics problem?

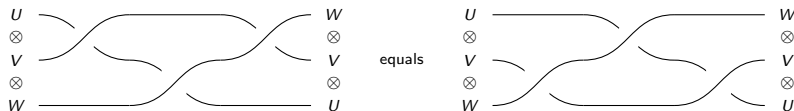
Braided monoidal categories

Definition

A **braided monoidal category** $\mathcal{C}$ consists of:

- Objects and morphisms between them ($\mathbb{C}$ -linear category).
- with a tensor product $X \otimes Y$ and a unit object 1
- with an associativity morphism $(X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$
- with a commutativity morphism $X \otimes Y \xrightarrow{\sim} Y \otimes X$

The commutativity morphism is called **braiding**, because it satisfies



Note that the double braiding is maybe not the identity.

In this sense the braid group $\mathbb{B}_n$ replaces the symmetric group $\mathbb{S}_n$.

Examples of braided monoidal categories

Example (Graded vector spaces)

Let Γ be a finite abelian group.

We have the following braided monoidal category $\text{Vect}_{\Gamma}^{\omega, \sigma}$

- 1-dimensional simple objects $\mathbb{K}_a$ for any $a \in \Gamma$.
- Tensor product $\mathbb{K}_a \otimes \mathbb{K}_b = \mathbb{K}_{ab}$.
- Associator $(\mathbb{K}_a \otimes \mathbb{K}_b) \otimes \mathbb{K}_c \xrightarrow{\omega(a,b,c)} \mathbb{K}_a \otimes (\mathbb{K}_b \otimes \mathbb{K}_c)$
- Braiding $\mathbb{K}_a \otimes \mathbb{K}_b \xrightarrow{\sigma(a,b)} \mathbb{K}_b \otimes \mathbb{K}_a$

for suitable pairs of functions (σ, ω) , called abelian 3-cocycles.

These are classified up to equivalence by quadratic forms Q on Γ .

Examples of braided monoidal categories

Example

The category of $\mathbb{C}$ -representations of a finite group G , with tensor product $\otimes_{\mathbb{C}}$, with trivial associator and braiding

Example

The category of $\mathbb{C}$ -representations of a Lie group G resp. Lie algebra $\mathfrak{g}$, with tensor product $\otimes_{\mathbb{C}}$, with trivial associator and braiding.

Examples of braided monoidal categories

Example (Drinfeld and Jimbo 1980's)

The category of representations of the **quantum group** $U_q(\mathfrak{g})$, which is a deformation of $U(\mathfrak{g})$ by a formal parameter q .

This category is semisimple, trivial associator, nontrivial braiding.

Drinfeld-Kohno theorem:

- This braiding describes the monodromy of the solutions of the Knizhnik-Zamolodchikov differential equation $KZ(\hat{\mathfrak{g}})$.

Kazhdan-Lusztig correspondence:

- This braided tensor category is equivalent to certain representations of the affine Lie algebra $\hat{\mathfrak{g}}$ at generic level k with $q = e^{\frac{\pi i}{k+h^\vee}}$ with the so-called vertex algebra tensor product involving analysis.

Examples of braided monoidal categories

Example (Lusztig 1990's)

For q an ℓ -th root of unity there is a non-semisimple quotient $u_q(\mathfrak{g})$.
Representations form a finite non-semisimple braided monoidal category.

Lusztig's motivation: Representations of $\mathfrak{g}$ over a field of characteristic ℓ .

Question

*Is there a Kazhdan-Lusztig type correspondence for $u_q(\mathfrak{g})$?
What about other braided tensor categories, can we "add analysis"?*

How can we construct $U_q(\mathfrak{g})$ and $u_q(\mathfrak{g})$
in a categorical way from Vect_{Γ} ?

Motivation



Nichols algebras

A **Nichols algebra** $\mathfrak{B}(X)$ is a universal (Hopf-)algebra attached to any object X in any braided monoidal category $\mathcal{C}$.

It encodes very nontrivial braid group combinatorics.

It appears as universal nilpotent algebra in the classification program for Hopf algebras (Andruskiewitsch-Schneider, Heckenberger, Angiono et al.)

Nichols algebras

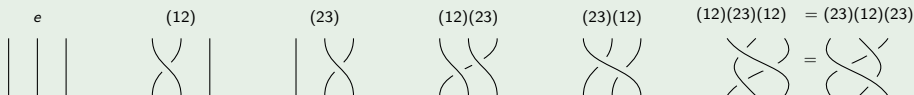
Definition (Matsumoto section)

Take the obvious homomorphism from braids to permutations $\mathbb{B}_n \rightarrow \mathbb{S}_n$. This does not split as groups, but as sets there is a section $s : \mathbb{S}_n \leftarrow \mathbb{B}_n$:

It maps the transposition $(i, i + 1)$ to the overcrossing braidings $c_{i,i+1}$, and maps **any** reduced expression to the respective product of braidings.

Example (n=3)

The Matsumoto section maps each element of $\mathbb{S}_3$ to the following braid:



Nichols algebras: Definition

Let X be any object in a braided monoidal category $\mathcal{C}$.

The braiding gives an action ρ_k of the braid group $\mathbb{B}_k$ on $X^{\otimes k}$ for any k .

Definition (Nichols '78, Woronowicz '89, Lusztig '93)

The Nichols algebra $\mathfrak{B}(X)$ is the tensor algebra $\bigoplus_{k \geq 0} X^{\otimes k}$ modulo the kernel of the **quantum symmetrizer map**

$$\mathbb{I}_{c,k} := \sum_{\tau \in \mathbb{S}_k} \rho_k(s(\tau)),$$

For example $\mathbb{I}_{c,2} = \text{id} + c$.

One can check that this is an algebra.

Nichols algebras: Example

As smallest example, take $X = x\mathbb{K}$ with braiding $x \otimes x \mapsto q(x \otimes x)$, then

$$\text{III}_{q,2}(x \otimes x) = (1 + q)(x \otimes x)$$

$$\begin{aligned}\text{III}_{q,3}(x \otimes x \otimes x) &= (1 + q + q + q^2 + q^2 + q^3)(x \otimes x \otimes x) \\ &= (1 + q)(1 + q + q^2)(x \otimes x \otimes x)\end{aligned}$$

$$\begin{aligned}\text{III}_{q,k}(x \otimes \cdots \otimes x) &= \left(\sum_{\sigma \in \mathbb{S}_k} q^{\text{length}(\sigma)} \right) (x \otimes \cdots \otimes x) \\ &= (1 + q) \cdots (1 + q + \cdots + q^k)(x \otimes \cdots \otimes x)\end{aligned}$$

So the Nichols algebra is $\mathcal{B}(X) = \mathbb{K}[x]/x^\ell$ if $1 + q + q^2 + \cdots + q^{\ell-1} = 0$ while otherwise we have $\mathcal{B}(X) = \mathbb{K}[x]$.

Nichols algebras: Examples

Example (Symmetric braiding)

For $c^2 = \text{id}$ the braided commutators $(\text{id} - c)(X \otimes X)$ are contained in the kernel of $\text{III}_2 = \text{id} + c$. Such Nichols algebras are braided commutative.

In more general cases, the Nichols algebra exhibits Serre-type relations.

Example

Fix a semisimple finite dimensional Lie algebra $\mathfrak{g}$ with simple roots α_i , root lattice Λ and Killing form (α_i, α_j) . For any $q \in \mathbb{C}^\times$ choose the braiding

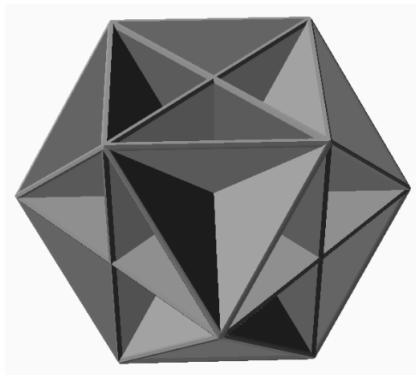
$$E_i \otimes E_j \mapsto q^{(\alpha_i, \alpha_j)}(E_j \otimes E_i)$$

Then the Nichols algebra $\mathfrak{B}(X)$ of the Λ -graded vector space X spanned by $E_1, \dots, E_n$ produces the nilpotent part of the (small) quantum group.

Nichols algebras: Root systems

Nichols algebras have a **generalized root system**.

The finite root systems can be classified in terms of hyperplane arrangements with an integrality condition [Heckenberger-Cuntz 2015]
5 series, 74 exceptionals in rank $3 \leq r \leq 8$.



A root system in rank 3 with 7 roots.
It may be familiar from the Lie superalgebra $D(2, 1|\alpha)$

Nichols algebras: An example beyond Vect_F

There exists a semisimple braided monoidal category $\mathcal{C}_v^u$ for $u, v \geq 2$ coprime integers. It has simple objects

$$X_0, X_1, \dots, X_{v-2}$$

The tensor product is a truncated version of $\mathfrak{sl}_2$ and cannot be realized by vector spaces. Associator and braiding are nontrivial. For example

$$X_1 \otimes X_1 = X_0 \oplus X_2$$

and braiding given by scalars $-s^{-3}$ on X_0 and s on X_2 , where $s = e^{\frac{\pi i}{2} \frac{u}{v}}$.

Nichols algebras: An example beyond Vect_Γ

Let us consider a Deligne product category $\mathcal{C}_v^u \boxtimes \text{Vect}_\Gamma$ such that for some $\alpha \in \Gamma$ the braiding on $\mathbb{C}_\alpha \otimes \mathbb{C}_\alpha$ is $-s^{-1}$. Then

$$X_1 \mathbb{C}_\alpha \otimes X_1 \mathbb{C}_\alpha = X_0 \mathbb{C}_{2\alpha} \oplus X_2 \mathbb{C}_{2\alpha}$$

with braiding given by the scalars $(-s^{-3})(-s^{-1}) = e^{-2\pi i \frac{u}{v}}$ and -1 .

Nichols algebras: An example beyond Vect_Γ

With these choices the Nichols algebra has relations in degree 2 and 3

$$\mathfrak{B}(X_1\mathbb{C}_\alpha) = X_0\mathbb{C}_0 \oplus X_1\mathbb{C}_\alpha \oplus X_0\mathbb{C}_{2\alpha} \oplus X_2\mathbb{C}_{2\alpha} \oplus 2X_1\mathbb{C}_{3\alpha} \oplus X_3\mathbb{C}_{3\alpha}$$

Remark

$\mathcal{C}_v^u$ can be constructed as a semisimplification of the representation category of the quantum group $U_q(\mathfrak{sl}_2)$ at $q = e^{2\pi i \frac{u}{v}}$.

X_1 is roughly the $\mathfrak{sl}_2$ standard representation and $\mathfrak{B}$ its exterior algebra.

Nichols algebra and tensor categories

Definition (Lusztig '93, Heckenberger '06, Andruskiewitsch Schneider '10,...)

The **Nichols algebra** $\mathfrak{B}(X)$ is the smallest Hopf algebra in $\mathcal{C}$ generated by all $x \in X$ with coproduct $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Example

Question: When is the square of a derivation x again a derivation?

Nichols algebra and tensor categories

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Example

Question: When is the square of a derivation x again a derivation?

$$(x \otimes 1 + 1 \otimes x)^2 = (x^2 \otimes 1) + (1 + q)(x \otimes x) + (1 \otimes x^2)$$

Answer: In characteristic 2 or in presence of a braiding $q = -1$ (fermion).

The Nichols algebra sets all higher derivations to zero
(and then allows controlled extensions, compare p -restricted Lie algebras)

Application

Representations of $\mathfrak{B}(X)$ in $\mathcal{C}$ produce a **nonsemisimple monoidal category**.

From Borel parts to full quantum groups

Application

Because a Nichols algebra $\mathfrak{B}(X)$ has the structure of a Hopf algebra, its representations inside $\mathcal{C}$ form a **nonsemisimple monoidal category**.

The relative Drinfeld center $\mathcal{Z}_{\mathcal{C}}$ of this monoidal category $\text{Rep}(\mathfrak{B}(X))$ containing $\mathcal{C}$ produces a **nonsemisimple braided monoidal category**.

Example

This gives a categorical construction of $\text{Rep}(u_q(\mathfrak{g}))$ from the category of weight spaces Vect_{Γ} and the Nichols algebra for the braiding $q^{(\alpha_i, \alpha_j)}$.

[Drinfeld89], [Majid91], [Gaiitsgory18], [GelakiNaiduNikshych09], [LaugwitzWalton22].

It contains many variants of quantum groups depending on the choice of $\mathcal{C}$ (simply connected, unrolled, multiparameter, quasi quantum group)

This also produces quantum super groups and more exotic examples.

From Borel parts to full quantum groups

Example (beyond Vect_Γ)

The relative Drinfeld center of the category of representations of the Nichols algebra $\mathfrak{B}(X_1 \mathbb{C}_\alpha)$ inside $\mathcal{C}_V^u \boxtimes \text{Vect}_\Gamma$ is **what?**

It looks like a "partially semisimplified" version of $u_q(\mathfrak{gl}(2|1))\text{-Rep}$.
More precisely: Pick the triangular decomposition $\mathfrak{gl}(2|1) = X_1 \mathfrak{gl}_2 X_1^*$ into even part $\mathfrak{gl}_2$ and odd part X_1 . Then semisimplify $\mathfrak{sl}_2$ in the zero part.
So this new modular tensor category is not so new after all.

Many more Nichols algebras in semisimplified quantum group categories can be constructed in such parabolic situations.

Question

Compare this construction to the classification of all abelian tensor ideals in Coulembier-Etingof-Ostrik and the list in Victors talk.

We finally sketch for this talk:

What kind of representation theory?

What kind of analysis or physics problem?

Schechtmann-Varchenko Integrals

Definition

Given a set of parameters $m_i, m_{ij} \in \mathbb{C}$ for $1 \leq i, j \leq n$ such that $m_{ij} = m_{ji}$ and any formal monomial $x_{i_1} \cdots x_{i_k}$ of degree k in n variables, we define this k -fold contour integral of a multivalued analytic function (exists iff...)

$$F := \int \cdots \int_{[e^0, e^{2\pi i}]^k} \prod_t z_t^{m_{i_t}} \prod_{i < j} (z_t - z_s)^{m_{i_t, i_s}} dz_1 \dots dz_k$$

Such integrals are used by Schechtman-Varchenko to construct solutions of $KZ(\hat{\mathfrak{g}})$ from $KZ(\hat{\mathfrak{h}})$ and geometrically explain Kazhdan-Lusztig corresp.

From the definition of Nichols algebras via quantum symmetrizers we find

Theorem (L. 2017)

A linear combination in F vanishes if the corresponding element in the Nichols algebra $\mathcal{B}(X)$ for braiding $x_i \otimes x_j \mapsto e^{\pi i m_{ij}}(x_j \otimes x_i)$ vanishes.

Are there similar integrals for other braided tensor categories?

Appendix: Example Degree 2

In degree 2 we can explicitly calculate F using the Euler Beta function B :

$$F(m_1, m_2, m_{12}) = \frac{e^{2\pi i m_2} - 1}{2\pi i} \frac{e^{2\pi i m_1 + 2\pi i m_{12}} - 1}{2\pi i} \frac{1}{m_1 + m_2 + m_{12} + 2} \cdot \left(B(m_2 + 1, m_{12} + 1) + \frac{\sin \pi m_1}{\sin \pi(m_1 + m_{12})} B(m_1 + 1, m_{12} + 1) \right)$$

$$F(m_1, m_2, m_{12}) = \tilde{F}(m_1, m_2, m_{12}) + e^{\pi i m_{12}} \cdot \tilde{F}(m_2, m_1, m_{12})$$

We can see the two types of quadratic Nichols algebra relations:

- for m_{12} an integer: $F(m_1, m_2, m_{12}) - (-1)^{m_{12}} F(m_2, m_1, m_{12}) = 0$
- for $m_1 = m_2$ and for m_{12} an odd integer: $F(m_1, m_1, m_{12}) = 0$.

Careful, where the formula has poles:

- $m_{12} \in -\mathbb{N}$, then the Nichols relation fails.
- $m_1 + m_2 + m_{12} + 2 = 0$, corresponding to Nichols algebra reflection.

Screening operators

A categorical upgrade of the previous statement says, very roughly:

- A vertex algebra is a commutative algebra depending on a variable z .
- There is a category of representations, with the vertex algebra tensor product and a braiding that expresses monodromy around $z = 0$.
- For every fixed object $\mathcal{M}$ there is a linear map called **screening**

$$\mathfrak{Z}_{\mathcal{M}} : \mathcal{N} \rightarrow \mathcal{M} \otimes \mathcal{N} \quad \text{for all } \mathcal{N}$$

Conjecture (L.)

Screenings $\mathfrak{Z}_{\mathcal{M}_1}, \dots, \mathfrak{Z}_{\mathcal{M}_n}$ generate the Nichols algebra $\mathfrak{B}(\mathcal{M}_1 \oplus \dots \oplus \mathcal{M}_n)$ inside this braided tensor category, provided all integrals exist.

For free field theories, where the braided tensor category is Vect_{Γ} , this follows from my statement on Schechtman-Varchenko integrals [L.17].

Big picture

I believe the Kazhdan-Lusztig correspondence is part of a general picture:

Conjecture (L.)

$$\begin{array}{ccc} \text{Vertex algebra Rep} & \xlongequal{\hspace{2cm}} & \mathcal{C} \\ \downarrow \text{Kernel of screenings} & & \downarrow \text{Add Nichols algebra} \\ \text{Vertex subalgebra Rep} & \xlongequal[\text{new Kazhdan-Lusztig}]{} & \mathcal{Z}_{\mathcal{C}}(\text{Rep}(\mathfrak{B})) \end{array}$$

where the same Nichols algebras appear on both sides,
as I proved for $\mathcal{C} = \text{Vect}_{\Gamma}$ for Schechtman-Varchenko integrals.

We have a general strategy to prove this equivalence using this and purely algebraic reconstruction results, that are new even for quantum groups.

Big picture

My main motivation was the **logarithmic Kazhdan Lusztig correspondence** proposed by B. Feigin and collaborators for small quantum groups ~ 2005 .



With T. Creutzig and M. Rupert we proved this for $u_q(\mathfrak{sl}_2)$, parallel to a very different proof by Gannon-Negron. I recently proved this for $u_q(\mathfrak{g})$ and more general diagonal braidings.

Big picture: Kazhdan Lusztig reproven

Recent work with T. Creutzig goes back to the roots:

We do **not** get back original Kazhdan-Lusztig correspondence.

We get **more** (but technical issues due to infinite dimension)

$$\begin{array}{ccc} \text{VirasoroRep} & \xlongequal{\quad} & u_q(\mathfrak{sl}(2)) \otimes u_{\bar{q}}(\mathfrak{sl}(2))\text{Rep} \\ \downarrow \text{Kernel of screenings} & & \downarrow \text{Add Nichols algebra} \\ \mathfrak{sl}_2\text{Rep, generic } \kappa & \xlongequal{\quad ! \quad} & u_q(\mathfrak{sl}(2|1)) \otimes u_{\bar{q}}(\mathfrak{sl}(2))\text{Rep} \end{array}$$

Here $\mathcal{C}$ is not a free field theory, and not semisimple.

The Nichols algebra is the odd part of $u_q(\mathfrak{sl}(2|1))$ over $u_q(\mathfrak{sl}_2)$.

Big picture: Beyond Vect_Γ

In current work with T. Creutzig we computed with these methods the category of weight modules of $\widehat{\mathfrak{sl}}_2$ at admissible level $-2 + \frac{u}{v}$.

$$\begin{array}{ccc}
 \text{Virasoro Rep} & \xlongequal{\quad\quad\quad} & u_q(\mathfrak{sl}(2)) \otimes u_{\bar{q}}(\mathfrak{sl}(2)) \text{Rep}^{ss} \\
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 \end{array}$$

Here the first row is not a free field theory but $\mathcal{C} = \mathcal{C}_v^u \boxtimes \mathcal{C}_u^v \boxtimes \text{Vect}_\Gamma$,
 The Nichols algebra and quantum group is the new one discussed above.

We have conjectures for $\widehat{\mathfrak{sl}}_n$ involving quantum supergroups. But why...?

This severely extends the Kazhdan-Lusztig correspondence

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This severely extends the Kazhdan-Lusztig correspondence
and triggered a very recent joint proof of the Fomin-Kirillov conjecture

Proof: Algebraic reconstruction results

The main thing given to us is that the first row $\mathcal{C}$ is equal the category $\mathcal{D}_A^{\text{loc}}$ of local modules over a commutative algebra A in the second row $\mathcal{D}$.

Example

The dual Verma module $\mathbb{V}_0^*$ is a commutative algebra A in $u_q(\mathfrak{g})\text{-Rep}$.
 A -modules with $\otimes_A$ are a monoidal category equivalent to $u_q(\mathfrak{g})^{\geq 0}\text{-Rep}$.
Local A -modules are a braided subcategory equivalent to $u_q(\mathfrak{g})^0\text{-Rep}$.

Can we revert this process?

Proof: Algebraic reconstruction results

We developed general algebra tools to reconstruct $\mathcal{D}$ from $\mathcal{D}_A^{\text{loc}}$



Proof: Algebraic reconstruction results

We developed general algebra tools to reconstruct $\mathcal{D}$ from $\mathcal{D}_A^{\text{loc}}$...and information on Ext^1 in $\mathcal{D}_A$ leading to a Nichols algebra in $\mathcal{D}_A^{\text{loc}}$.

More precisely, under technical assumptions (detailed slides below)

- $\mathcal{D}$ is necessarily a relative Drinfeld center of $\mathcal{D}_A$.
- $\mathcal{D}_A$ contains a subcategory of objects with local composition factors, which can be realized by modules over some Hopf algebra $\mathfrak{B}$ in $\mathcal{D}_A^{\text{loc}}$. In our case all *simple* modules are local, so this subcategory is all.
- Using deep results from Nichols algebra theory (diagonal braiding or by-hand), we show that $\mathfrak{B}$ is indeed the Nichols algebra

$$\mathfrak{B}\left(\bigoplus_i \text{Ext}^1(1, X_i)X_i\right)$$

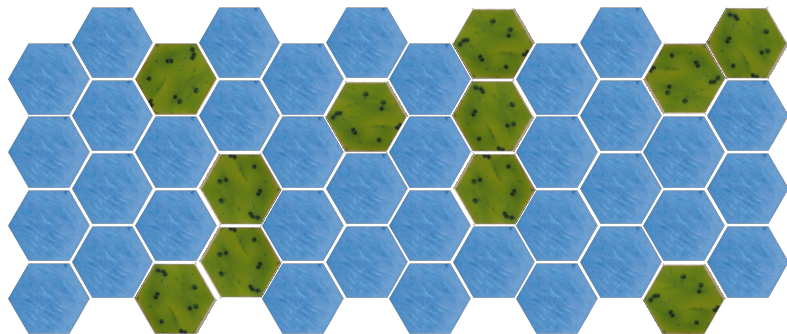
where X_i runs over all simple local modules.

Towards quantum field theory: A problem

Fill a rectangle with a hexagonal lattice of side length ϵ .

Color each hexagon randomly with probability p .

Consider the limit of a very fine lattice $\epsilon \rightarrow 0$.



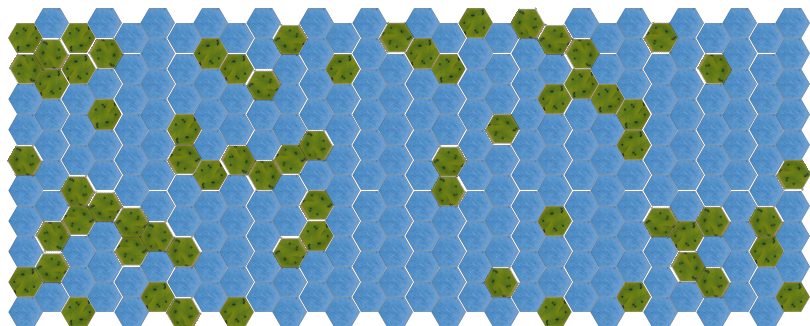
$$p = 25\%$$

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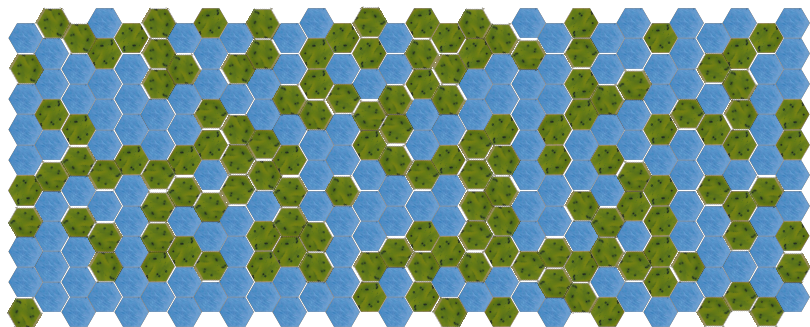
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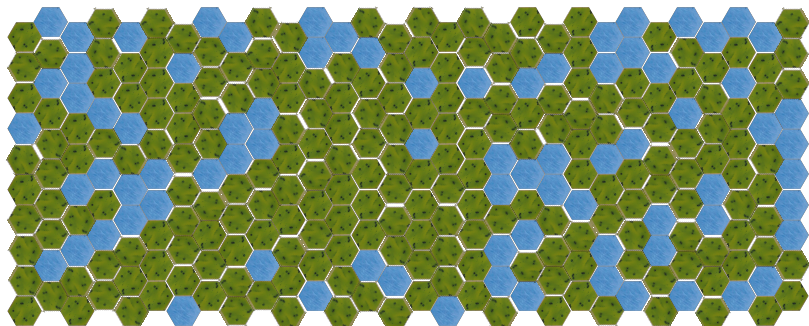
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Towards quantum field theory: A problem

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$$p = 75\%$$

Towards quantum field theory: A problem

At $p_{crit} = 50\%$ the behavior changes from **islands** to **continent**.

We can now ask questions in our random model, e.g.

With what probability $\mathbb{P}(p)$ exists a grass path left to right edge?

Towards quantum field theory: A problem

At $p_{crit} = 50\%$ the behavior changes from **islands** to **continent**.

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Cardy (1992) conjectured from quantum field theory that independent of the lattice we have

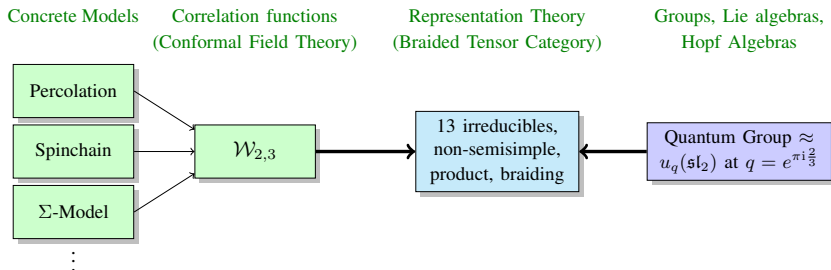
$$\mathbb{P}(p) = \begin{cases} 0 & \text{for } p < p_{crit} \\ \frac{\Gamma(2/3)}{\Gamma(1/3)\Gamma(4/3)} z^{1/3} \cdot {}_2F_1(z) & \text{for } p = p_{crit} \\ 1 & \text{for } p > p_{crit} \end{cases}$$

with z a crossratio and ${}_2F_1(z)$ the hypergeometric function.

Smirnov (2001) proved this for hexagonal lattice, $p_{crit} := 50\%$.

A random example: Percolation

This and other observables of critical percolation are manifestation of a conformal quantum field theory $\mathcal{W}_{2,3}$



Such complicated objects also have a representation theory that describes dimensions, magic numbers, product decomposition.

Appendix: Reconstruction theorem 1

Theorem (Creutzig-L.-Rupert 2023)

Let A be a commutative algebra in a braided monoidal category $\mathcal{D}$. Assume $\mathcal{D}$ and $\mathcal{D}_A$ are finite and rigid, and the braiding in $\mathcal{D}$ nondeg.

Then $\mathcal{D}$ is equivalent to the Drinfeld center of the monoidal category $\mathcal{D}_A$, relative to the braided subcategory $\mathcal{D}_A^{\text{loc}}$.

- The equivalence is by the induction functor $A \otimes (-)$, which produces an A -module with a central structure. This is based on [Schauenburg01].

Appendix: Reconstruction theorem 2

We have the following relative version of Tannaka-Krein reconstruction:

Theorem (L.-Mombelli 2023)

*Let $\mathcal{B}$ be a finite and rigid monoidal category
and $\mathcal{C} \subset \mathcal{B}$ be a central braided monoidal category.*

*Assume this inclusion **splits** by an exact faithful tensor functor F*

$$\mathcal{C} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow[F \quad]{} \end{array} \mathcal{B}$$

*Then there exists a Hopf algebra $B \in \mathcal{C}$ such that $\mathcal{B} \cong \text{CoRep}(B)$,
and the functor F is the functor forgetting the B -action
and the inclusion of $\mathcal{C}$ is given by the trivial B -action.*

- If $\mathcal{C}$ is the category of representations of a Hopf algebra C , then the result is a version of Radfords projection theorem.

Appendix: Reconstruction theorem 2b

Using ideas of the Andruckiewitsch-Schneider program, and in particular Heckenberger's classification of Nichols algebras over Vect_Γ^Q and results in [Angiono13], we can determine B as follows:

Theorem (L. 2025)

Let $\mathcal{B}$ be a finite and rigid monoidal category and $\mathcal{C} \subset \mathcal{B}$ be a central braided monoidal category, equivalent to Vect_Γ^Q , such that all simple object in $\mathcal{B}$ are in $\mathcal{C}$.

Then $\mathcal{B} \cong \text{CoRep}(B)$ with B the Nichols algebra of the following $\mathcal{C}$ -object

$$\bigoplus_{a \in \Gamma} \text{Ext}_{\mathcal{B}}^1(\mathbb{C}_a, 1) \mathbb{C}_a$$

Appendix: Reconstruction theorem 1+2

Taking $\mathcal{B} \subset \mathcal{D}_A$ the A -modules with composition series in $\mathcal{C}$ we get:

Corollary (L. 2025)

Suppose $\mathcal{D}$ is a finite rigid braided monoidal category, and A is a commutative algebra in $\mathcal{D}$ such that $\mathcal{D}_A^{\text{loc}} = \text{Vect}_\Gamma^Q$.

Then a subquotient of $\mathcal{D}$ is the generalized quantum group associated to $\mathcal{C}$ and the Nichols algebra B in the previous slide.

In particular, if Frobenius Perron dimensions of them match, then $\mathcal{D}$ equivalent to the generalized quantum group.