

SPECIAL RELATIVITY HOMEWORK – WEEK 2

Exercise 1. In the lecture, we described the “relativistic Hamiltonian for a non-relativistic system”:

$$\mathcal{H}(t, p_t, q^I, p_I) = p_t + H(t, q^I, p_I) , \quad (1)$$

where H is the standard Hamiltonian. Write Hamilton’s equations of motion for t and p_t , treating them the same as the other coordinates and momenta. What do these equations mean?

Exercise 2. A symmetric matrix $A_{ij} = A_{ji}$ can be decomposed as:

$$A_{ij} = a\delta_{ij} + B_{ij} , \quad (2)$$

where B_{ij} is called the traceless part of A_{ij} , satisfying $B_{ii} = 0$. Find the explicit expressions for a and B_{ij} .

Exercise 3. Consider a totally symmetric rank-3 tensor $A_{ijk} = A_{(ijk)}$ (make sure you understand this equation). Denote its trace as $A_i \equiv A_{ijj}$. Find an expression for the traceless part of A_{ijk} .

Exercise 4. In elementary physics, there are three main “axial vectors”, i.e. vectors that rely on the right-hand rule: angular velocity $\boldsymbol{\omega}$, angular momentum $\mathbf{J}$, and the magnetic field $\mathbf{B}$. We can get rid of the right-hand rule by converting them into antisymmetric matrices $\omega_{ij}, J_{ij}, B_{ij}$, related to the vectors as $\omega_i = \frac{1}{2}\epsilon_{ijk}\omega_{jk}$ etc. Rewrite the following equations using index notation and the matrices $\omega_{ij}, J_{ij}, B_{ij}$, without any ϵ_{ijk} ’s in the final result.

1. Rotational velocity: $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.

2. Angular momentum: $\mathbf{J} = \mathbf{r} \times \mathbf{p}$.

3. Magnetic force: $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$.

4. Maxwell’s equations:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0 ; \quad \boldsymbol{\nabla} \times \mathbf{E} = -\dot{\mathbf{B}} ; \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0 ; \quad \boldsymbol{\nabla} \times \mathbf{B} = \mu_0(\mathbf{j} + \epsilon_0\dot{\mathbf{E}}) . \quad (3)$$