

SPECIAL RELATIVITY HOMEWORK – WEEK 1

An infinitesimal symmetry transformation can be written as $g = 1 + \epsilon G$, where ϵ is an infinitesimal parameter, and G is called a generator. Multiplying $1 + \epsilon G$ by itself a large number of times $1/\epsilon$, we get the general/finite symmetry transformation:

$$g = (1 + \epsilon G)^{1/\epsilon} = e^G . \quad (1)$$

Exercise 1. Consider an infinitesimal transformation $g = 1 + \epsilon G$.

1. Suppose g is a matrix that satisfies $gMg^T = M$, for a fixed matrix M . What is the corresponding condition on the generator G ? Hint: work to 1st order in ϵ .
2. Now suppose g satisfies $gMg^\dagger = M$. What is the corresponding condition on G ?
3. Now suppose g satisfies $\det g = 1$. What is the corresponding condition on G ? Hint: work with the definition of the determinant as a sum of products of matrix elements. Only one of these products contributes to 1st order in ϵ .

Exercise 2. Consider two group elements $g = e^G$ and $h = e^H$. Their non-commutativity can be “measured” via the product hgh^{-1} . Let us show that this can be reduced to commutators of generators, such as $[H, G]$, $[H, [H, G]]$ etc. Taylor expansions are your friend!

1. Show that hgh^{-1} can be written as $e^{hGh^{-1}}$.
2. Show that hGh^{-1} can be written as:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\left[H, [H, \dots, [H, G]] \dots \right]}_{n \text{ commutators}} \equiv e^{[H, \cdot]} G . \quad (2)$$