

(Super)symmetric polynomials and the Pieri rule

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Motivation

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In this talk, we will present a special case of this multiplication when all coefficients are equal to 1.

Partitions and Young diagrams

A **partition** $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ of n is a non-increasing sequence of integers such that $\sum_{i=1}^m \lambda_i = n$. We write $\lambda \vdash n$ if λ is a partition of n .

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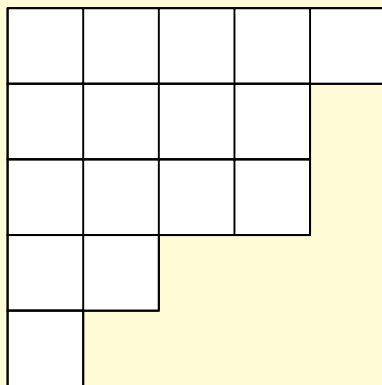
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For example, take $n = 16$ and $\lambda = (5, 4, 4, 2, 1)$. The Young diagram is drawn below.



Tableaux

If we fill the boxes of the Young diagram with integers, we obtain a **tableau of shape** λ . We call a tableau **semi-standard** if entries weakly increase along the rows and strictly increase down the columns. If they also strictly increase along the rows, we say that the tableau is **standard**.

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$$T_1 = \begin{array}{|c|c|c|c|} \hline 2 & 3 & 1 & 4 \\ \hline 6 & 7 & 8 & \\ \hline 3 & 3 & & \\ \hline 5 & 1 & & \\ \hline \end{array}$$
$$T_2 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 4 & 4 & 5 & \\ \hline 5 & 8 & & \\ \hline 6 & 11 & & \\ \hline \end{array}$$
$$T_3 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \hline 8 & 9 & & \\ \hline 10 & 11 & & \\ \hline \end{array}.$$

Then T_2 is semi-standard and T_3 is standard.

Row insertion

The algorithm **row insertion** takes a tableau T and inserts a positive integer x into it, resulting in a new tableau, denoted $T \leftarrow x$.

1. If $x \geq i$ for all entries i in the first row of T , add x to the end of the first row.
2. If not, find the leftmost i in the first row of T such that $i > x$.
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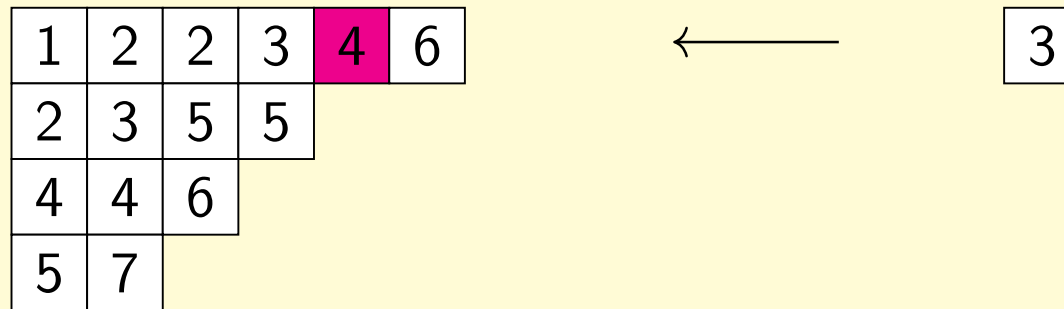
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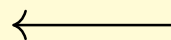
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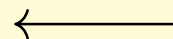
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Row bumping lemma

Assume that we successively insert two integers a and b (in this order) into some T . We label the resulting new boxes in T by B_a and B_b . Then we have the following:

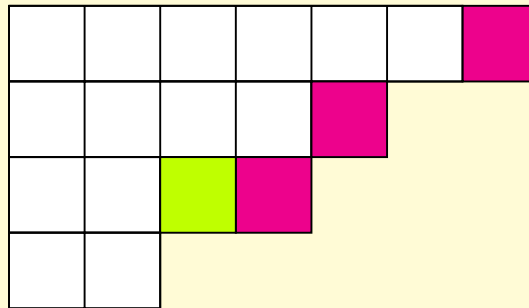
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If $a \leq b$ and B_a is the green node, then the possible positions of B_b are highlighted in pink.

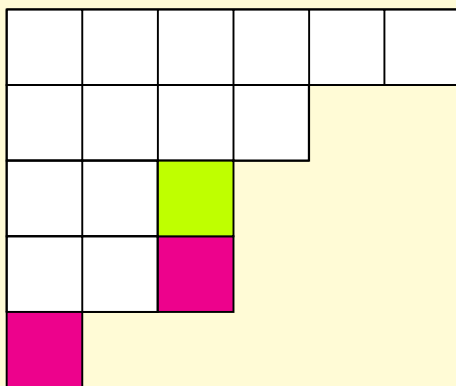


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Similarly, if $a > b$ and B_a is the green node, then the possible positions of B_b are highlighted in pink.



Symmetric polynomials

Let $\mathbf{x} = (x_1, x_2, \dots, x_\ell)$ be a set of variables. The monomial $x_1^{a_1} x_2^{a_2} \dots x_\ell^{a_\ell}$ is said to have **degree n** if $\sum_i a_i = n$.

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The **monomial symmetric polynomial corresponding to λ** is given by

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For example, if $\lambda = (2, 1)$ and $\mathbf{x} = (x_1, x_2, x_3)$, then

$$m_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2.$$

The n th elementary symmetric polynomial ($n \leq \ell$) is given by

$$e_n = m_{(n)} = \sum_{i_1 < \dots < i_n} x_{i_1} \dots x_{i_n}$$

and the n th complete homogeneous symmetric polynomial is given by

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For example, if $n=4$ and $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$, then

$$e_4 = x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_5 + x_1 x_2 x_4 x_5 + x_1 x_3 x_4 x_5 + x_2 x_3 x_4 x_5,$$

$$h_4 = x_1^4 + x_2^4 + \dots + x_1^3 x_4 + x_1^3 x_3 + \dots + x_1 x_2 x_4 x_5 + \dots.$$

The polynomial e_n is the sum of all square-free monomials of degree n .

Schur polynomials

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$$s_\lambda(\mathbf{x}) = s_\lambda(x_1, \dots, x_\ell) = \sum \mathbf{x}^T$$

where the sum is taken over all monomials coming from **semi-standard T** of shape λ filled with numbers from 1 to ℓ .

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Continuing with the previous example, if

$$T_2 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 4 & 4 & 5 & \\ \hline 5 & 8 & & \\ \hline 6 & 11 & & \\ \hline \end{array}, \quad \text{then } \mathbf{x}^{T_2} = x_1 x_2 x_3 x_4^2 x_5^2 x_6^2 x_8 x_{11}.$$

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Observe, that if $\lambda = (n)$ (a single row) or $\lambda = (1^n)$ (a single column), then

$$s_{(n)}(\mathbf{x}) = h_n(\mathbf{x}) \quad \text{and} \quad s_{(1^n)}(\mathbf{x}) = e_n(\mathbf{x}).$$

Pieri rule

The following is an immediate consequence of the row bumping lemma:

Corollary (Pieri rule)

Using the insertion process from before, we obtain the following formulas:

$$s_{\lambda} s_{(n)} = \sum_{\mu} s_{\mu}$$

where the sum is taken over all μ 's that are obtained from λ by adding n nodes, with no two in the same column; and

$$s_{\lambda} s_{(1^n)} = \sum_{\mu} s_{\mu}.$$

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Example of the Pieri rule

Take $\lambda = (4, 3, 2, 2)$ and $\nu = (3)$. Then using the formula from the previous slide, we see that

$$s_\lambda s_\nu = s_{(4,3,2,2)} s_{(3)} = \sum_{\mu} s_\mu$$

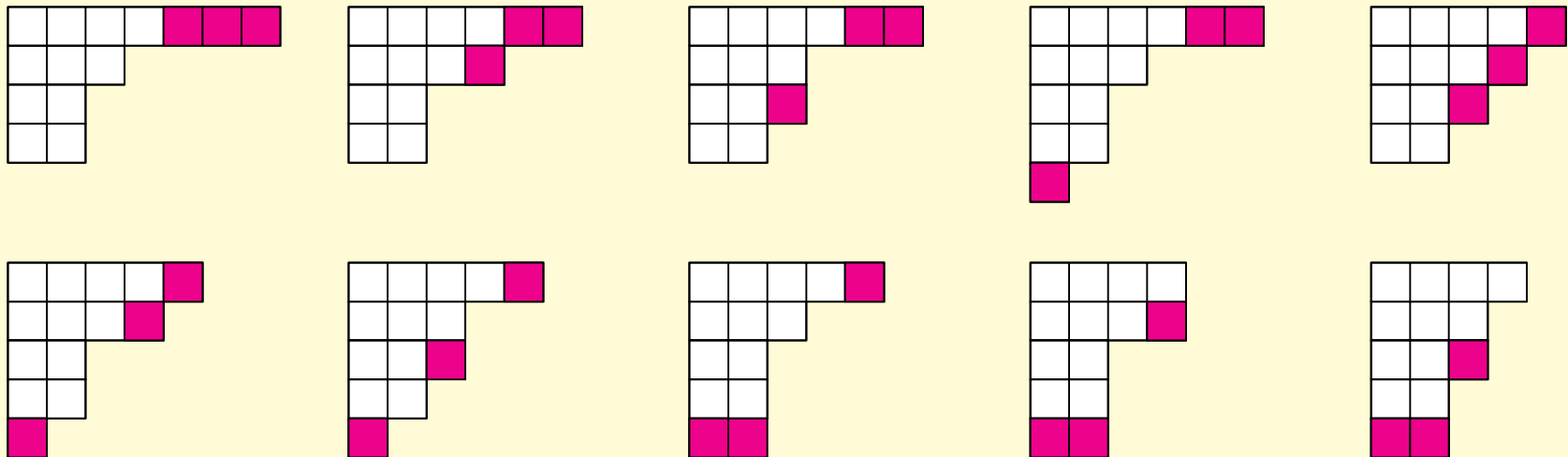
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where μ is obtained from λ by adding 3 boxes, no two in the same column. We see that the sum consists of the Schur polynomials corresponding to the following tableaux:



Supersymmetric polynomials

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ be a partition of n and $\mathbf{x} = (x_1, x_2, \dots, x_\ell)$ and $\mathbf{y} = (y_1, y_2, \dots, y_k)$ two sets of variables.

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A polynomial $p(\mathbf{x}, \mathbf{y})$ is called **supersymmetric** if it is symmetric in both \mathbf{x} and \mathbf{y} and upon substitution $x_1 = t$ and $y_1 = -t$ the resulting expression is independent of t .

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For example, $x_1 + x_2 + x_3 + y_1 + y_2$ is supersymmetric (in $3 + 2$ variables).

Supersymmetric Schur polynomials

Characters of the Lie superalgebra $\mathfrak{gl}_{(m|n)}$.

Just like Schur polynomials, we can read them off from **semi-standard** supertableaux:

$$S_\lambda(\mathbf{x}/\mathbf{y}) = \sum \mathbf{x}^T \mathbf{y}^T.$$

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Entries in a supertableau have **even or odd parity** such that

$$1' < 2' < 3' < \dots < k < 1 < 2 < \dots < \ell.$$

Odd (primed) entries must strictly increase along the rows and even (unprimed) entries must strictly increase down the columns.

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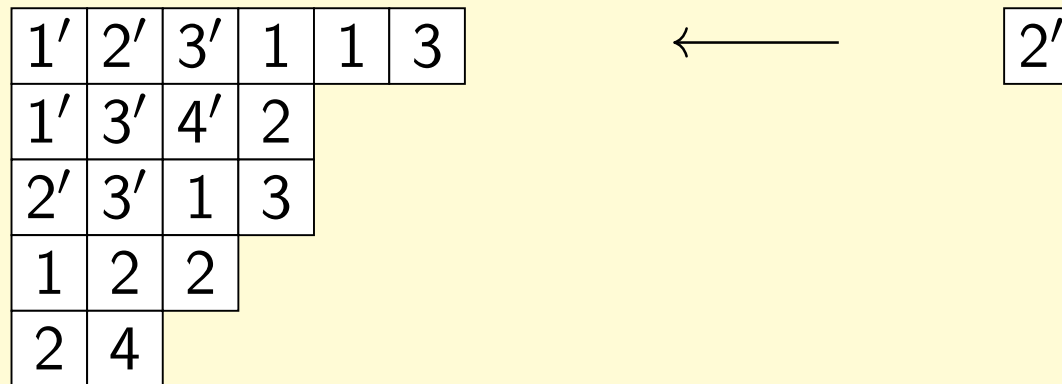
For example, if $\lambda = (6, 4, 4, 3, 2)$ and

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1' & 2' & 3' & 1 & 1 & 3 \\ \hline 1' & 3' & 4' & 2 & & \\ \hline 2' & 3' & 1 & 3 & & \\ \hline 1 & 2 & 2 & & & \\ \hline 2 & 4 & & & & \\ \hline \end{array}, \quad \text{then } \mathbf{x}^T = y_1^2 y_2^2 y_3^3 y_4 x_1^4 x_2^4 x_3^2 x_4.$$

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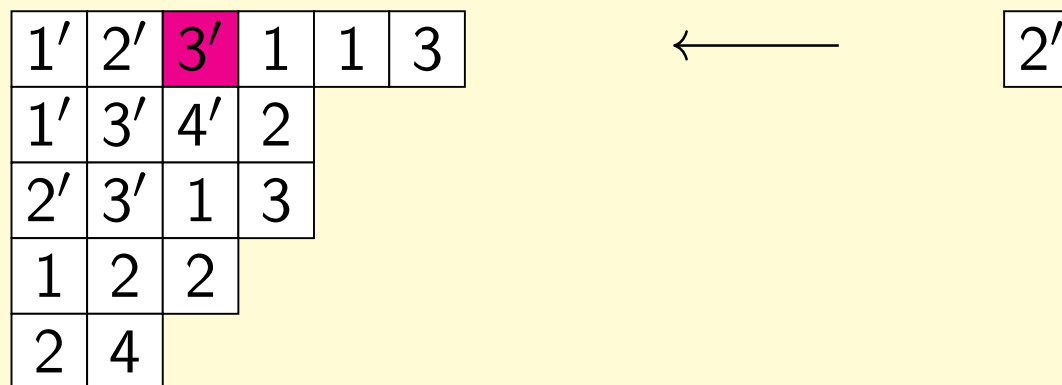
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1'	3'	4'	2				
2'	3'	1	3				
1	2	2					
2	4						

We see that simple row-insertion does not preserve the semi-standardness of our tableau.

Mixed insertion

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Two different algorithms:

- ε -insertion: even numbers are inserted in the next row down and odd numbers are inserted in the next column to the right,

Mixed insertion

In his paper, Muth described a new 'mixed' insertion process, called ε -insertion, which takes into account the parities of the entries in T , by defining ε to be even or odd.

Two different algorithms:

- **even-insertion**: even numbers are inserted in the next row down and odd numbers are inserted in the next column to the right,
- **odd-insertion**: odd numbers are inserted in the next row down and even numbers are inserted in the next column to the right.

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In order to preserve the semi-standardness of T , we modify the second step in row-insertion. Recall that

2. If not, find the leftmost i in the first row of T such that $i > x$.

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In order to preserve the semi-standardness of T , we modify the second step in row-insertion. Recall that

2. If not, find the **leftmost/uppermost** i in the **first row/column** of T such that $i > x$ (even-ins.) or $i \geq x$ (odd-ins.).

Example of even-insertion

Under even insertion, even numbers are inserted in the next row down and odd numbers are inserted in the next column to the right, and we are looking for entries i such that $i > x$.

1'	2'	3'	1	1	3
1'	3'	1	2		
2'	3'	2	3		
1	3	3			
2					



1'

Example of even-insertion

Under even insertion, even numbers are inserted in the next row down and odd numbers are inserted in the next column to the right, and we are looking for entries i such that $i > x$.

1'	2'	3'	1	1	3
1'	3'	1	2		
2'	3'	2	3		
1	3	3			
2					



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1'	2'	3'	1	1	3
1'	3'	1	2		
1'	3'	2	3		
1	3	3			
2					



2'

Example of even-insertion

Under even insertion, even numbers are inserted in the next row down and odd numbers are inserted in the next column to the right, and we are looking for entries i such that $i > x$.

1'	2'	3'	1	1	3
1'	3'	1	2		
1'	3'	2	3		
1	3	3			
2					



2'

Example of even-insertion

Under even insertion, even numbers are inserted in the next row down and odd numbers are inserted in the next column to the right, and we are looking for entries i such that $i > x$.

1'	2'	3'	1	1	3
1'	2'	1	2		
1'	3'	2	3		
1	3	3			
2					



3'

Example of even-insertion

Under even insertion, even numbers are inserted in the next row down and odd numbers are inserted in the next column to the right, and we are looking for entries i such that $i > x$.

1'	2'	3'	1	1	3
1'	2'	1	2		
1'	3'	2	3		
1	3	3			
2					

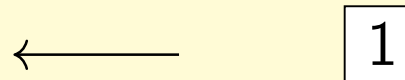


3'

Example of even-insertion

Under even insertion, even numbers are inserted in the next row down and odd numbers are inserted in the next column to the right, and we are looking for entries i such that $i > x$.

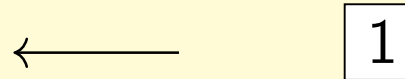
1'	2'	3'	1	1	3
1'	2'	3'	2		
1'	3'	2	3		
1	3	3			
2					



Example of even-insertion

Under even insertion, even numbers are inserted in the next row down and odd numbers are inserted in the next column to the right, and we are looking for entries i such that $i > x$.

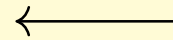
1'	2'	3'	1	1	3
1'	2'	3'	2		
1'	3'	2	3		
1	3	3			
2					



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Under even insertion, even numbers are inserted in the next row down and odd numbers are inserted in the next column to the right, and we are looking for entries i such that $i > x$.

1'	2'	3'	1	1	3
1'	2'	3'	2		
1'	3'	1	3		
1	3	3			
2					

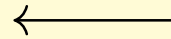


2

Example of even-insertion

Under even insertion, even numbers are inserted in the next row down and odd numbers are inserted in the next column to the right, and we are looking for entries i such that $i > x$.

1'	2'	3'	1	1	3
1'	2'	3'	2		
1'	3'	1	3		
1	3	3			
2					

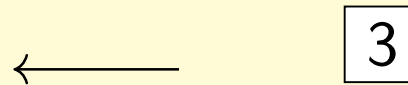


2

Example of even-insertion

Under even insertion, even numbers are inserted in the next row down and odd numbers are inserted in the next column to the right, and we are looking for entries i such that $i > x$.

1'	2'	3'	1	1	3
1'	2'	3'	2		
1'	3'	1	3		
1	2	3			
2					



Example of even-insertion

Under even insertion, even numbers are inserted in the next row down and odd numbers are inserted in the next column to the right, and we are looking for entries i such that $i > x$.

1'	2'	3'	1	1	3
1'	2'	3'	2		
1'	3'	1	3		
1	2	3			
2	3				

Result

Corollary (The Pieri formulas for supersymmetric Schur polynomials)

Following the ε -insertion as defined above, we obtain the following formulas:

$$S_\lambda S_{(n)} = \sum_{\mu} S_\mu$$

where the sum is taken over all μ 's that are obtained from λ by adding n nodes, with no two in the same column; and

$$S_\lambda S_{(1^n)} = \sum_{\mu} S_\mu.$$

where the sum is taken over all μ 's that are obtained from λ by adding n nodes, with no two in the same row.