Satake's Good Basic Invariants for Finite Reflection Groups

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March 20, 2023@OIST

Symmetric Polynomials

Let S_n be the symmetric group of degree n.
S_n acts on $\mathbb{R}[x_1, \ldots, x_n]$ by permuting the variables.
A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is symmetric if f is invariant under the S_n-action.

Example

The Elementary Symmetric Functions:

$$E_1 = \sum_{i=1}^{n} x_i, \quad E_2 = \sum_{1 \leq i < j \leq n} x_i x_j, \quad E_3 = \sum_{1 \leq i < j < k \leq n} x_i x_j x_k, \quad \ldots$$

Theorem (Fundamental thm of Symmetric Functions)

$E_1, \ldots, E_n$ are algebraically independent, and

$$\mathbb{R}[x_1, \ldots, x_n]^{S_n} = \mathbb{R}[E_1, \ldots, E_n] .$$

This theorem is generalized to finite reflection groups.

Finite Reflection Groups

Let V be a Euclidean space of dimension n.

Definition

1. A reflection is a linear transformation s which sends some nonzero vector to its negative while fixing the hyperplane orthogonal to the vector. Such a vector is called a root of the reflection.
2. A finite subgroup of $GL(V)$ generated by reflections is called a finite reflection group.

Irreducible finite reflection groups were classified, and they are called

$$A_n(\cong S_{n+1}), B_n, D_n, E_6, E_7, E_8, F_4, G_2, H_3, H_4, I_2(m) \ .$$

Basic Invariants

Let G be a finite reflection group acting on a Euclidean space V of dimension n.
Denote by S the algebra of polynomial functions on V. The G-action on V induces an actiton on S. An element $g \in G$ acts on $f \in S$ by

$$(gf)(v) = f(g^{-1}v) \quad (v \in V) \; .$$

$f \in S$ is G-invariant if $gf = f$ holds for all $g \in G$.
The subalgebra of G-invariant polynomials is denoted S^G.

Theorem (Chevalley 1955)

S^G is generated by n homogeneous algebraically independent polynomials of positive degrees.

Such a set of generators is called a set of basic invariants. The degrees $d_1, \ldots, d_n$ of generators $f_1, \ldots, f_n$ are uniquely determined by G. We assume that $d_1 \leq d_2 \leq \ldots \leq d_n$.

Example: A_{n-1}

Define the linear action of S_n on $\mathbb{R}^n$ by

$$\sigma \boldsymbol{e}_i = \boldsymbol{e}_{\sigma(i)} \quad (1 \leq i \leq n)$$

where $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is the standard basis of $\mathbb{R}^n$.

1. Each transposition (i, j) acts as the reflection w.r.t. the hyperplane $x_i = x_j$ where $x_1, \ldots, x_n$ are the coordinates associated to $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$.
2. The S_n-action fixes the line $\mathbb{R}(\boldsymbol{e}_1 + \cdots + \boldsymbol{e}_n)$. Thus S_n acts on the orthogonal complement $V = \{x_1 + \cdots + x_n = 0\}$ and this action is irreducible.

S_n together with its action on V is called the finite reflection group of type A_{n-1}. For A_{n-1},

$$S = \mathbb{R}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n), \quad S^G = \mathbb{R}[E_2, \ldots, E_n]\ ,$$

and the degrees of A_{n-1} are $2, 3, \ldots, n$.

Remark

The choice of a set of basic invariants is not unique in general (even if up to constant).
For example, for A_{n-1}, you can take the power sums

$$P_\alpha = \sum_{i=1}^{n} x_i^\alpha \quad (2 \leq \alpha \leq n)$$

as a set of basic invariants.
So it is natural to ask whether there exists a “canonical” choice of basic invariants. This problem was studied by Saito–Yano–Sekiguchi in 1980.

History

In 1980, Saito–Sekiguchi–Yano wrote in an article:

> *So far, however, there has seldom been any attempt to distinguish one system of generators from any other. The main purpose of this article is to show that there exists a uniquely specified generator system* $f_1, \ldots, f_n$ *for the ring* S^G *(up to constant factors) by adding a certain condition on* $f_1, \ldots, f_n$.
>
> ...
>
> *One may ask whether one can find a generator system* $f_1, \ldots, f_n$ *such that* $\frac{\partial}{\partial f_n}(\langle df_i, df_j \rangle)_{i,j}$ *is a constant matrix.*

Here $\langle \; , \; \rangle$ denotes a metric on the cotangent bundle TV^* induced from the Euclidean inner product on V.

Such a set of basic invariants is called a set of flat invariants.

1. In the article, Saito–Yano–Sekiguchi proved the uniqueness of a set of flat invariants for irreducible finite reflection groups.
2. They also showed the existence by explicitly constructing flat invariants except E_7, E_8.
3. For E_7, a set of flat invariants was constructed by Yano in 1981.
4. The existence for all irreducible finite reflection groups was proved by Saito in an article published in 1993.

Example: calculation of flat invariants for A_3

Since the degrees of A_3 are $2, 3, 4$, up to constant mutiple, basic invariants must be of the form

$$f_1 = E_2, \quad f_2 = E_3, \quad f_3 = E_4 + cE_2^2 \ .$$

If we impose Saito–Yano–Sekiguchi's condition

$$\frac{\partial}{\partial f_3}(\langle df_i, df_j \rangle)_{i,j} = \text{a constant matrix},$$

we obtain

$$c = -\frac{1}{8} \ .$$

Satake's Good Basic Invariants

In 2020, Satake proposed a notion of good basic invariants which are defined using a Coxeter element. He proved that good basic invariants are flat invariants.
To explain his definition, I recall the root system, the simple system and the Coxeter element.

Root System and Simple Sysetm

Let G be an irreducible finite reflection group acting on a Euclidean space V of dimension n.

A root system Φ of G is a finite subset of V and it is constructed as follows. For each reflection $s \in G$, take a root α_s, and consider the set

$$\Phi = \{\pm\alpha_s \mid s \text{ is a reflection in } G\}.$$

Here, the lengths of the roots must be chosen so that $G(\Phi) = \Phi$.

A simple system Δ is a subset of a root system Φ satisyfing:

- Δ is a basis of V,
- Each $\alpha \in \Phi$ is either a nonnegative linear combination of Δ or a nonpositive linear combination of Δ.

A simple system exists and any two simple systems are conjugate under G.

G is generated by reflections corresponding to the roots in Δ.

Coxeter Elements

Given a simple system Δ, a Coxeter element is constructed as follows. Enumerate elements of Δ as $\alpha_1, \ldots, \alpha_n$.
Let $s_1, \ldots, s_n \in G$ be the corresponding reflections. Then

$$g = s_1 \cdots s_n$$

is called a Coxeter element.
Any two Coxeter elements are conjugate under G.
The order of a Coxeter element is called the Coxeter number of G and it is equal to the highest degree d_n.

Properties of Coxeter elements

Take a Coxeter element g of an irreducible finite reflection group G and let $h(= d_n)$ be the Coxeter number. To deal with eigenvectors of g, we consider the complexification $V_{\mathbb{C}}$ of V.

Theorem

1. *g has a primitive h-th root of unity ζ as an eigenvalue. The eigenspace is one-dimensional and eigenvectors are regular (i.e. do not lie on any reflection hyperplanes).*
2. *N eigenvalues of g are ζ^{1-d_α}, where $d_1, \ldots, d_n$ are the degrees of G.*

Definition (Satake 2020)

A triple (g, ζ, q) is called an <u>admissible triplet</u>. Here $g \in G$ is a Coxeter element, $\zeta \in \mathbb{C}$ is an eigenvalue of g which is a primitive h-th root of unity and whose eigenvector $q \in V_{\mathbb{C}}$ is regular.

Example: A_3

A root system and a simple system:

$$\Phi = \{\pm(\boldsymbol{e}_i - \boldsymbol{e}_j) \mid 1 \le i < j \le 4\}, \quad \Delta = \{\boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{e}_2 - \boldsymbol{e}_3, \boldsymbol{e}_3 - \boldsymbol{e}_4\}$$

The reflections s_1, s_2, s_3 corresponding to the simple roots and the coxeter element $g = s_1 s_2 s_3$:

$$s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The order of g is four and the eigenvalues are $-i, -1, i, 1$ with eigenvectors

$$q_1 = \begin{pmatrix} i \\ -1 \\ -i \\ 1 \end{pmatrix}, \quad q_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad q_3 = \begin{pmatrix} -i \\ -1 \\ -i \\ 1 \end{pmatrix}, \quad q_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

The last one corresponds to the fixed line $\mathbb{R}(\boldsymbol{e}_1 + \cdots + \boldsymbol{e}_4)$ and hence is irrelevant to A_3. q_1, q_3 are regular (while q_2 is not). Therefore (g, i, q_3) is an admissible triplet.

Good Basic Invariants

Fix an admissible triplet (g, ζ, q) and take a basis $\{q_1, q_2, \ldots, q_n = q\}$ of $V_{\mathbb{C}}$ consisting of eigenvectors of the Coxeter element g with

$$g q_\alpha = \zeta^{1-d_\alpha} q_\alpha.$$

Let $z_1, \ldots, z_n$ be the associated linear coordinates of $V_{\mathbb{C}}$.
Set

$$I_\alpha = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0} \mid a_1 d_1 + \cdots + a_n d_n = d_\alpha, a_1 + \cdots + a_n \geq 2\}$$

for $1 \leq \alpha \leq n$.

Definition (Satake 2020)

A set of basic invariants $f_1, \ldots, f_n$ is good w.r.t. the admissible triplet (g, ζ, q) if $f_1, \ldots, f_n$ satisfy

$$\frac{\partial^a f_\alpha}{\partial z^a}(q) = 0 \quad (1 \leq \alpha \leq n, a \in I_\alpha)\ .$$

Here,

$$\frac{\partial^a}{\partial z^a} := \prod_{\beta=1}^{n} \frac{\partial^{a_\beta}}{\partial z_\beta^{a_\beta}}$$

for $a = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n$.

Theorem (Satake 2020)

1. *For a given admissible triplet, a set of good basic invariants exists.*
2. *The vector subspace of S^G spanned by a set of good basic invariants depends neither on the choice of admissible triplet, nor on the choice of coordinates $z_1, \ldots, z_n$.*
3. *A set of good basic invariants is flat.*

Example: A_3

The relationship between the standard coordinates of $\mathbb{R}^4$ (restricted to $V = \{x_1 + \cdots + x_4 = 0\}$) and the new coordinates z_1, z_2, z_3 associated to q_1, q_2, q_3 is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = z_1 q_1 + z_2 q_2 + z_3 q_3.$$

Substituting this into E_2, E_3, E_4, we have

$$E_2 = -2z_2^2 - 4z_1 z_3, \quad E_3 = 4z_1^2 z_2 + 4z_2 z_3^2,$$
$$E_4 = -z_1^4 + z_2^4 - 4z_1 z_2^2 z_3 + 2z_1^2 z_3^2 - z_3^4 \ .$$

Given that $d_1 = 2, d_2 = 3, d_3 = 4$, $I_1 = I_2 = \emptyset$ and $I_3 = \{(2,0,0)\}$. For $f_1 = E_2, f_2 = E_3, f_3 = E_4 + cE_2^2$ to satisfy the goodness condition,

$$\frac{\partial^2 f_3}{\partial^2 z_1^2}(q_3) = 4 + 32c = 0 \quad \therefore c = -\frac{1}{8}$$

Remarks

1. Kyoji Saito's flat structure contains not only flat invariants but also a product structure on TV. In 2020, Satake also found a formula expressing the product in terms of the good basic invariants and its derivatives.
2. Satake's definition of good basic invariants includes finite complex reflection groups. In that case, a Coxeter element must be replaced by a d_n-regular element.
3. In the joint work with Minabe, we showed the existence and the uniqueness of good basic invariants for duality groups, and also obtained a formula for the product (work in progress).

References

- Saito–Yano–Sekiguchi, On a certain generator system of the ring of invariants of a finite reflection group. Comm. Algebra 8 (1980),
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