Women at the Intersection of Mathematics and Theoretical Physics

March 2023, OIST

Black holes and the arithmetic of of families of Calabi-Yau varieties

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Women at the intersection of mathematics and theoretical physics
OIST, 20-24 March 2023

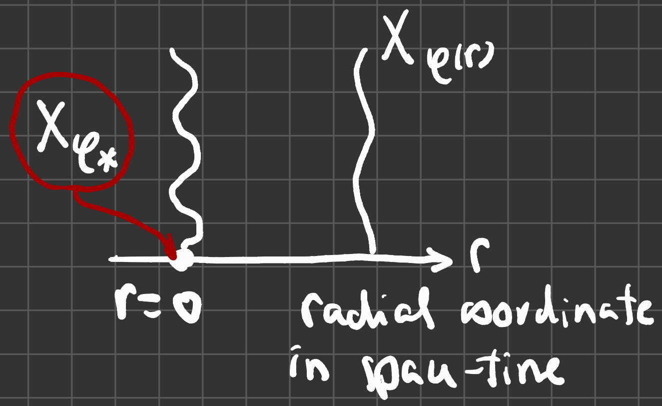
AIM

an instance of the rich connections between mathmatics (geometry & number theory) and theoretical physics geometry of Calabi-Yau varieties and that of their moduli spaces $\longleftrightarrow$ arithmetic of families of CY manifolds and modularity properties

string theory

focus today: physics of Black hole solutions in string theory

entropy
$\updownarrow$
L-values



attractor mechanism

work with

P Candelas, M. Elmi
& D van Straten
Dec 2019

P Candelas, XD, D van Straten April 2021

(P Candelas, XD, J Mc Govern, P Kuusela 2021, Feb 2023)

① CY manifolds — very brief

② Arithmetic of CY manifolds — mostly classical discussion

③ The attractor mechanism
- black hole solutions of IIB superstrings
- arithmetic and modularity

1 CALABI-YAU MANIFOLDS

Mathematical objects of interest:

$\underbrace{\text{algebraic varieties}}_{\text{set of solutions of}}$ with certain special properties

Calabi-Yau manifolds

$$\boxed{P(\varphi, \underline{x}) = 0, \quad \underline{x} \in \mathbb{A}}$$

polynomials with complex coefficients φ

metric $\sim \partial\bar{\partial}K$

CY manifolds: compact Kähler with $c_1=0$

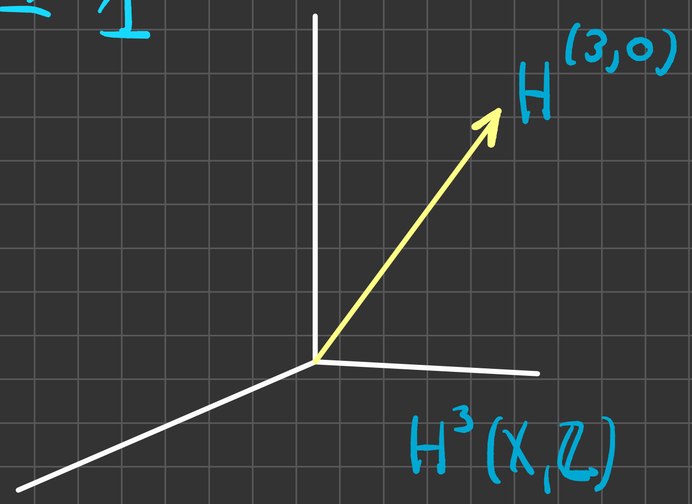
admit a Ricci-flat metric

This talk: concerned with $d=3$

- It is a theorem that $\exists!$ (up to a constant) $(3,0)$-form Ω which is holomorphic ($d\Omega=0$)

$$h^{(3,0)} = \dim H^{(3,0)} = \dim H^{(0,3)} = 1$$

$$\left(H^3 = H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)} \right)$$



▶ CY manifolds have parameters X_φ

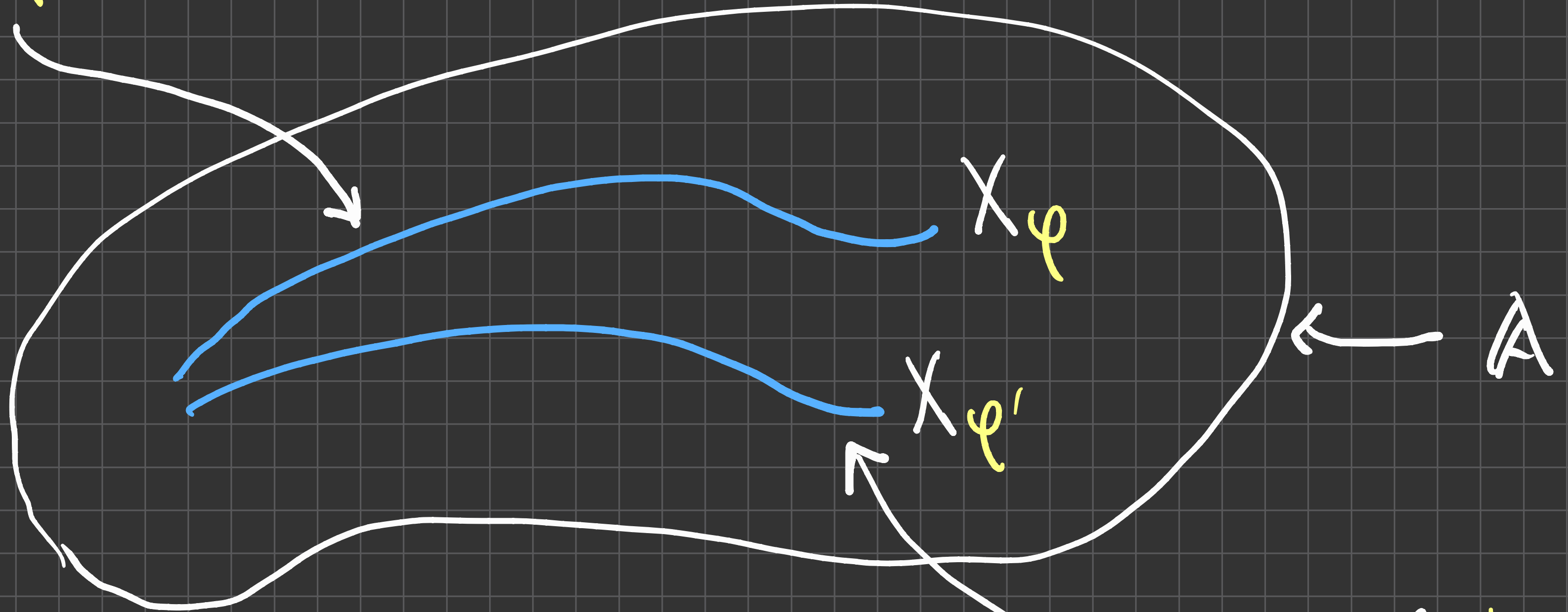
↳ they come in families

complex structure

Kähler structure

(moduli spaces have interesting geometry) complex structure parameters $\rightarrow$ coefficients of P

$X_\varphi : \; P(\varphi, \underline{x}) = 0$



$X_{\varphi'} : \; P(\varphi', \underline{x}) = 0$

Kähler structure: "size" with respect to the metric

Examples: very many

- $\mathbb{P}^4[5]$ eg $\sum x_i^5 - 5\psi x_1 x_2 x_3 x_4 x_5 \quad (x_1, \dots, x_5) \in \mathbb{P}^4$

  $c_1 = 0$

- We have in mind a particular example:

  Verrill 1996, Hulek & Verrill 2005

$$\left[\left(\sum_{i=1}^{5} x_i \right) \left(\sum_{i=1}^{5} \frac{1}{x_i} \right) - \varphi^{-1} = 0 \quad \text{when } (x_1, \dots, x_5] \in \mathbb{P}^4 \setminus \{ \textstyle\prod x_i = 0 \} \right]$$

why this example?

exhibits interesting arithmetic properties which have an interpretation in BH solutions of string theory

② ARITHMETIC OF CALABI-YAU VARIETIES

Let X_φ be a family of algebraic varieties st X_φ is a hypersurface with defining polynomial $P(\varphi, \underline{X})$

let $\varphi \in \mathbb{Q}$

Questions:

- how many solutions of $P(\underline{X}, \varphi) = 0$ are there over $\mathbb{Q}$

  ie $X_i \in \mathbb{Q}$

- how does this number vary with φ?

$\boxed{\text{TOO HARD}}$

eg in number theory

$\rightsquigarrow$ millenium problems BSD-conjecture

related to estimates of these countings

for elliptic curves

One learns a lot however by "reducing mod p" where p is a prime number, that is, by working over finite fields $\mathbb{F}_{p^k}$, $k = 1, 2, \dots$

$(\mathbb{F}, +, \times)$

field with p^k elements

[simplest: $\mathbb{F}_p \longrightarrow$ integers mod p

$\mathbb{F}_{p^2} \longrightarrow \mathbb{F}_p[\alpha] = \{ a + \sqrt{\alpha} b,\ a, b \in \mathbb{F}_p,\ \alpha \neq \text{a square in } \mathbb{F}_p \}$]

eg 2, 3 not squares in $\mathbb{F}_5$

So let $\varphi \in \mathbb{F}_p$:

The fundamental quantities of interest are

$N_k(\varphi)$ = number of solutions of $P(\underline{x}, \varphi) = 0$ over $\mathbb{F}_{p^k}$

Generating function $\rightsquigarrow$ zeta function

$$\zeta_X(T, p; \varphi) = \sum_{k=1}^{\infty} \frac{1}{k} N_k(\varphi) T^k$$

depends on both p & φ

properties $\rightsquigarrow$ Weil conjectures

(proven by Dwork, Deligne, Grothendieck) "simple" case : X a point

$$N_k = 1 \quad \forall k$$

$$\sum_{k=1}^{\infty} \frac{1}{k} N_k T^k = \sum_{k=1}^{\infty} \frac{1}{k} T^k = -\log(1-T)$$

$$\Rightarrow \quad \zeta_{point}(T) = \frac{1}{1-T}$$

Remark: $\quad \prod_p \zeta_{point}(p^{-s}) = \prod_p \frac{1}{1-p^{-s}}$

$$= \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta_{\mathbb{R}}(s)$$

enter the notion of L-functions

Example: Elliptic curve $\mathcal{E}$ CY dim 1

$$\zeta(T) = \frac{1 + a_p T + p T^2}{(1-T)(1-pT)} \qquad \boxed{a_p = 1 + p - N_1}$$

$p \neq$ prime of bad reduction

$\mathcal{E}$ associated to a modular form

Taniyama-Shimura conjecture

proof: Wiles, Breuil, Conrad, Diamond, Taylor

L-function: $$\text{"} \prod_p \zeta(p^{-s}) \text{"} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

$a_n \rightsquigarrow$ n-th coeff of a modular form of weight 2 of $\Gamma_0(N) < SL(2,\mathbb{Z})$

↳ built from the a_p

conductor: its prime factors → primes of bad reductions

$$\Gamma_0(N): \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \text{ st } c = 0 \bmod N$$

weight k modular form $g(\tau)$ of $\Gamma_0(N)$: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$

$$g\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k g(\tau)$$

For a smooth CY 3-fold X

$$\zeta_X(T) = \frac{\cancel{R_1(T)}\, R_3(T)\, \cancel{R_5(T)}}{(1-T)\, R_2(T)\, R_4(T)\, (1-p^3T)}$$

Weyl conjectures:
→ ζ is a rational function
→ $\deg R_i = b_i$

CY 3 fold: $b_1 = 0$, $b_5 = 0 \Rightarrow R_1 = 1$ & $R_5 = 1$

$\deg R_3(T) = b_3$; $\quad b_3 = \underset{h^{3,0}}{1} + \underset{h^{2,1}}{1} + \underset{h^{1,2}}{1} + \underset{h^{0,3}}{1} = 4$ $\quad$ 1 parameter examples

$$\underset{\substack{\| \\ b_2}}{\deg R_2(T)} = \underset{\substack{\| \\ b_4}}{\deg R_4(T)} = h^{11} \qquad (\text{as } h^{(2,0)} = h^{(0,2)} = 0)$$

$b_2 = b_4$

<u>nicest</u> cases → $R_2(T) = (1-pT)^{h^{11}}$, $R_4(T) = (1-p^2T)^{h^{11}}$

$h^{1,2} = 1$ so $b_3 = 1 + 1 + 1 + 1 = 4$

$$R_3(T, \varphi) = 1 + a_p(\varphi) T + b_p(\varphi) p T^2 + a_p(\varphi) p^3 T^3 + p^6 T^4$$

$$a_p(\varphi) = 1 + h^{11} p + h^{11} p^2 + p^3 - N_1(\varphi)$$

\# of points over $\mathbb{F}_p$

$$2p\, b_p(\varphi) = 1 + h^{11} p^2 + h^{11} p^4 + p^6 + N_1^2(\varphi) - N_2(\varphi)$$

\# of points over $\mathbb{F}_{p^2}$

These can be computed!

R_3 can be "quickly" computed for $\varphi \in \mathbb{F}_p$ ($\varphi = 0, 1, \dots, p-1$) for many primes p

P. Candelas, XD & D van Straten (04/2021)

Many questions arise: certainly ready for substancial experimentation.

→ What are the properties of a_p & b_p?

→ One can construct L-functions

$$L(s,\varphi) = \text{"}\prod_p R_3(p^{-s},\varphi)^{-1}\text{"}$$

What are the properties of the L-function?

Modularity conjectures?

That is: is there an analogue of the modularity of elliptic curves for CY 3-folds?

(⟶ Langlands programme)

Again at this time these are very **hard** questions

modularity is <u>not</u> classical modularity except in some special cases

- rigid CY $(h^{2,1} = 0)$: F. Gouvea + N. Yui

$$R_3(T) = 1 - a_p T + p^3 T^2$$

▶ What happens at singularities?

necessary to properly understand the L-function

eg conifold singularities

R_3 gets a factor $(1 - a_p T + p^3 T^2)$

a_p = p-th coeff in q-expansion of the eigenform g of weight 4 of $\Gamma_0(N)$

mirror quintic: $\varphi = 1/5$ $N = 25$

similar statements for HV

In fact

HV motivation : "to find further examples of modular
(2005)
CY varieties, ie, of CY varieties which are defined over the rationals and whose L-series can be described in terms of modular forms"

found modularity at conifold singularities

Later:

modularity at values of φ^* where X_{φ^*} is smooth
(attractor varieties)

Recall: a_p & b_p depend on φ

While at this time it is hard to say in general what the properties of a_p & b_p are (and there are many conjectures)

$\hookrightarrow$ interesting things happen for special values of φ

③ THE ATTRACTOR MECHANISM PART 1

(Ferrara, Kallosh, Strominger 95, ... Greg Moore 98 ....)

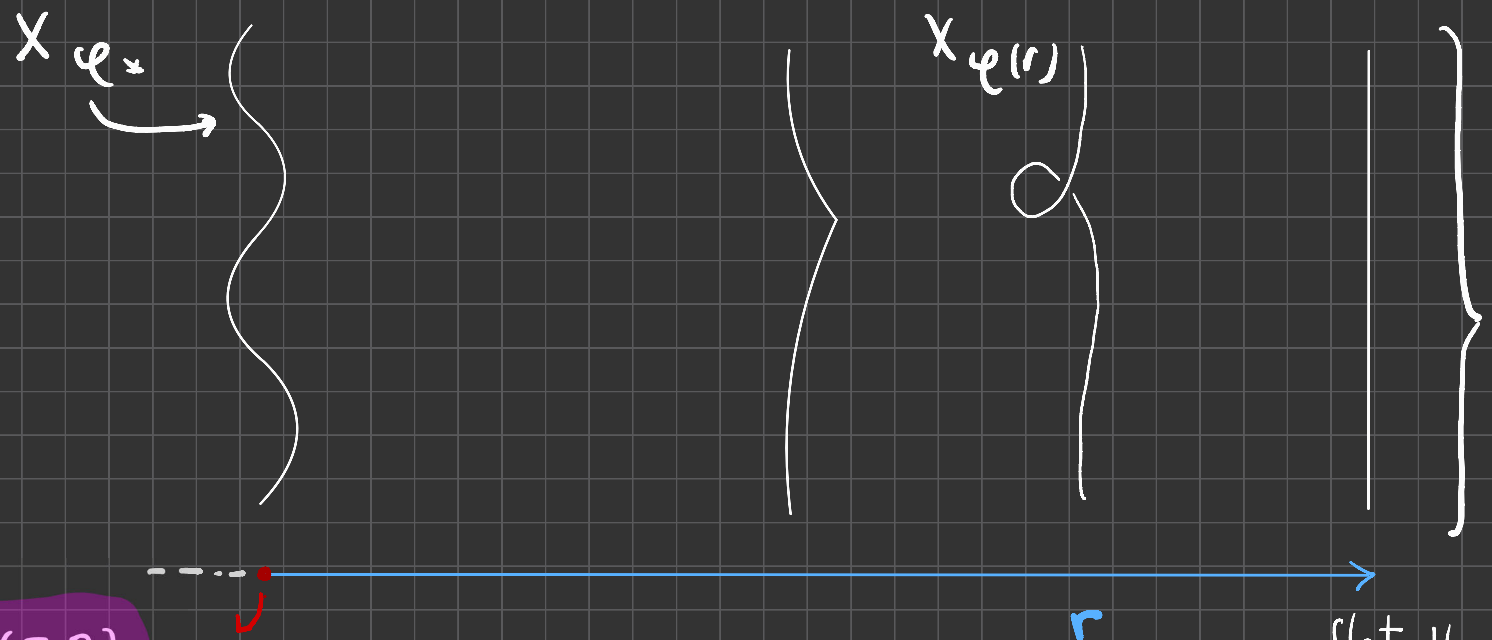
Physics : supersymmetric black hole solutions of type IIB supergravity

10 dimensional generalisations of Einstein's equations for gravity + Maxwell equations for electromagnetism

$$\text{10 dim space-time} \left\{ \begin{array}{c} \text{4 dim spherically symmetric, asymptotically flat, charged BH parametrised by a radial coordinate } r \\ \times \\ \text{6 dim CY } X_{\varphi(r)} \text{ at each point of the BH} \end{array} \right.$$

$$\text{10 dim metric} = \left(\begin{array}{c|c} \text{BH metric } (r) & 0 \\ \hline 0 & \begin{array}{c} \text{CY metric which} \\ \text{depends on } \varphi(r) \end{array} \end{array} \right)$$

10 dim space: a CY, $X_{\varphi(r)}$, at each point of space-time (BH)



CY with complex structure varies with r

horizon $(r=0)$

$U(r) \to -\infty$

(coordinate singularity)

- surface separating interior & exterior of the BH
- last surface visible from infinity

flat 4-dim space-time $r \to \infty$

$(U(r) \to 0)$

4 dim spherically symmetric BH

$$ds^2 = e^{2U(r)} dt^2 + e^{-2U(r)} d\underline{x}^2, \quad \underline{x} = (x, y, z)$$

Type IIB SUGRA is gravity with extra $u(1)$ gauge fields (b_3 of them)

So, the BH has electric & magnetic charge

$$Q = \begin{pmatrix} q_a \\ p^b \end{pmatrix} \qquad a, b = 0, \dots, h^{2,1}(X)$$

these are integers

let: $\Gamma := p^a \alpha_a - q_b \beta^b \in H^3(X, \mathbb{Z})$

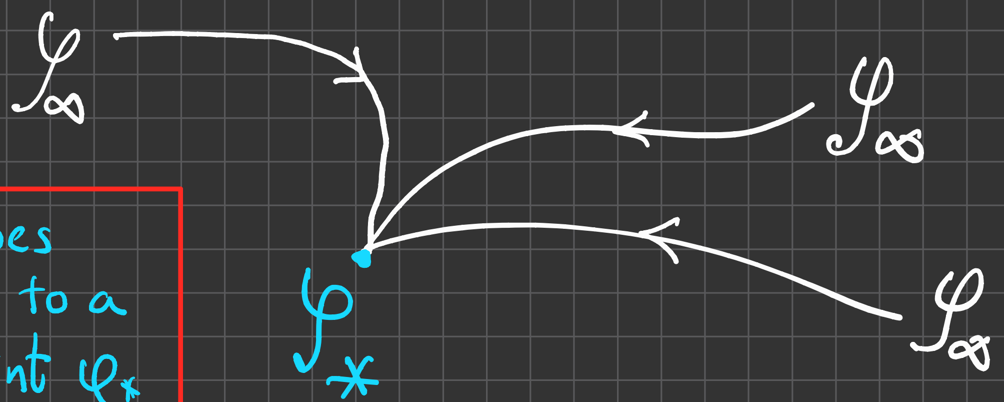
charge vector

(α_a, β^b) basis of $H^3(X, \mathbb{Z})$

Black hole solutions which preserve supersymmetry need to satisfy 1st order differential eqs for $t(r)$ & $\varphi(r)$, the attractor equations

These equations represent a non-linear dynamical system on the $\mathbb{C}$ structure moduli space with flow parameter $\rho = 1/r$

The attractor equations say that for a solution with $\Gamma \in H^3(X, \mathbb{Z})$, the $\mathbb{C}$-structure parameters flow to a value $\varphi_* = \varphi(r=0)$ independent of the starting value $\varphi_\infty = \varphi(r=\infty)$



φ evolves smoothly to a fixed point φ_* at $r=0$

95: Ferrara + Kallosh + Strominger

. . . . .

98: G. Moore conjectures on the arithmetic nature of attractor varieties X_{φ_*}

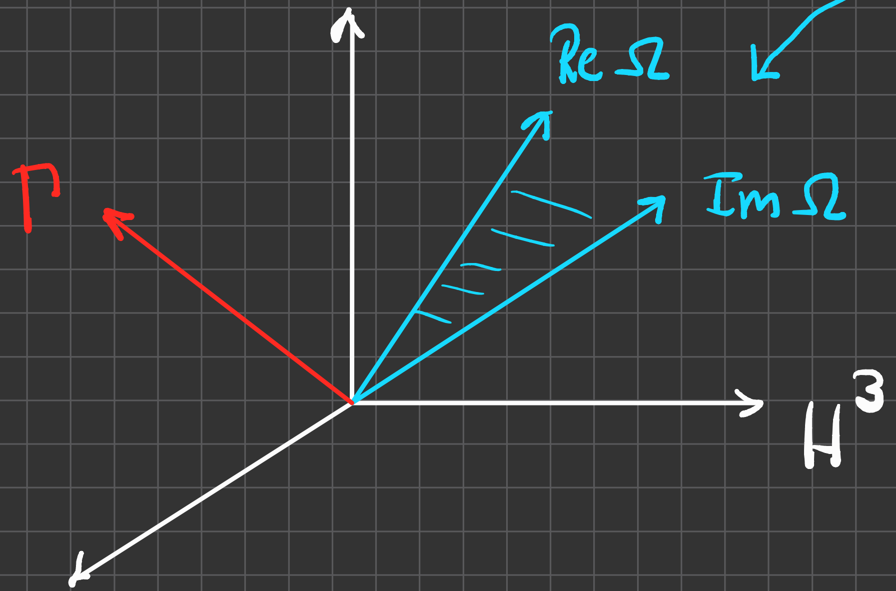
The $\mathbb{C}$-structure at an attractor point $\varphi = \varphi_*$ is st

$$\boxed{\begin{gathered} \Gamma = p^a \alpha_a - q_a \beta^a \in H^{(3,0)} \oplus H^{(0,3)} \\ \text{ie} \quad \Gamma^{(2,1)} = \Gamma^{(1,2)} = 0 \end{gathered}}$$

- one can solve the attractor eqs for charges Γ st φ_* is an attractor point but the result <u>generically</u> is that Γ is <u>not</u> integral

$\boxed{\text{rank } 1}$ attractors

Recall: Ω defines a line in $H^3(X,\mathbb{Z})$



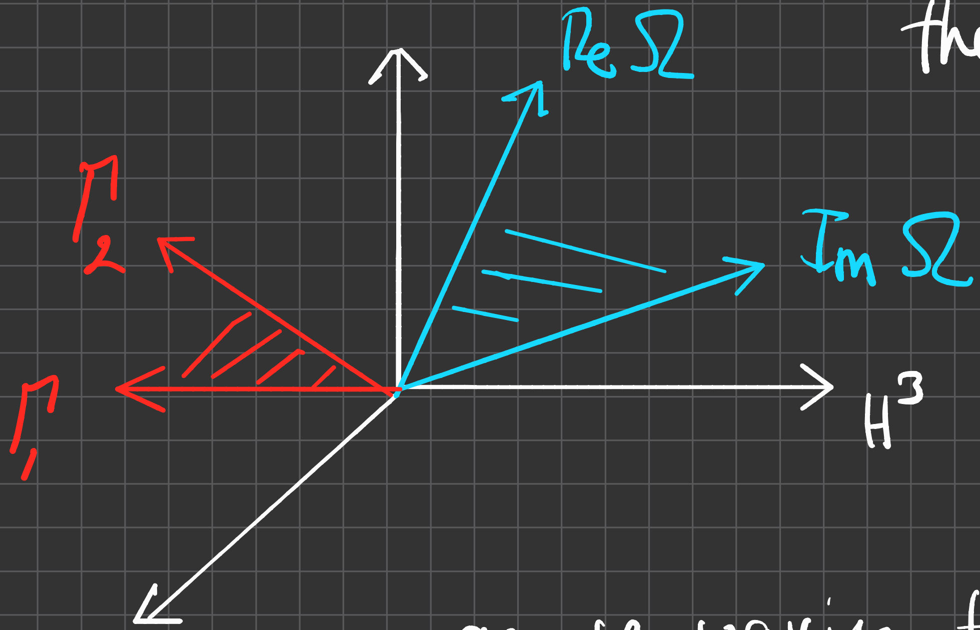
Consider $V_{\mathbb{R}}(\varphi)$ = plane spanned over $\mathbb{R}$ by Re Ω & Im Ω

$V_{\mathbb{R}}(\varphi)$ moves with φ

OTOH: Inside $H^3(X,\mathbb{R})$ we have a lattice of vectors $\Gamma \subset H^3(X,\mathbb{Z})$ which are fixed

rank 1 A.P: φ st $V_{\mathbb{R}}(\varphi)$ contains the line Γ

rank 2 attractors



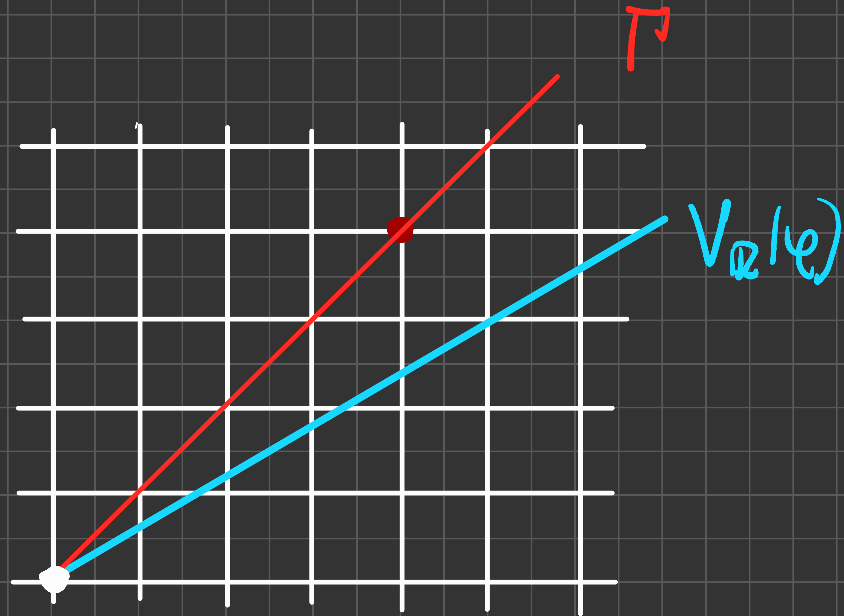
at an attractor point of rank 2 there are $\boxed{\text{two}}$ vectors in $H^3(X,\mathbb{Z})$

Γ_1, Γ_2 st $\Gamma_{1,2} \in H^{(3,0)} \oplus H^{(0,3)}$

as φ varies the plane $V_{\mathbb{R}}(\varphi)$ moves and at a <u>rank 2</u> attractor point $\varphi = \varphi_*$ the plane $V_{\mathbb{R}}(\varphi)$ <u>coincides</u> with the plane generated by Γ_1 & Γ_2

RARE, very difficult to find a CY which has rank 2 attractor points

A line which passes through the origin in general will <u>not</u> pass through another lattice point unless the slope is rational.



Not too hard to find φ st $V_{\mathbb{R}}(\varphi)$ coincides with Γ

For rank 2 attractors we have a plane and it is then much harder to find φ st it coincides with $V_{\mathbb{R}}(\varphi)$

(some progress ... P. Candelas + XD + M. Elmi + D van Straten
K Bönish + A Klemm + Scheidegger + Zagier
P. Candelas + XD + J McGovern + P Kuusela in progress ...)

Geometrically: at $\varphi = \varphi^*$

$V = H^{(3,0)} \oplus H^{(0,3)}$ is a lattice plane in $H^3(X,\mathbb{Z})$

Then $V^\perp = H^{(2,1)} \oplus H^{(1,2)}$ is orthogonal to V (under the natural symplectic product on 3-forms) and it is also a lattice plane in $H^3(X,\mathbb{Z})$

This amounts to a

$\therefore$ $\boxed{\text{splitting of the Hodge structure of } H^3(X,\mathbb{Z})}$

Hodge conjecture $\Rightarrow$ splitting has a geometrical origin

So, how do we find attractor varieties with rank 2 attractor points?

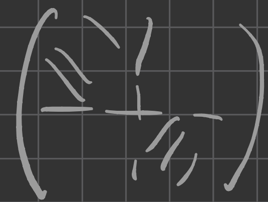
again, very hard!

However: the splitting becomes apparent in the arithmetic structure of X

$\therefore$ arithmetic strategy!

3 Part II The attractor mechanism & the arithmetic of CY manifolds

$\boxed{\text{At } \varphi = \varphi_*}$

$\mathrm{Frob}^{-1} \rightsquigarrow$ 

$$R_3(T) = \det(\mathbb{1} - T\,\mathrm{Frob}_p^{-1})$$

$$= (1 - p\alpha T + p^3 T^2)(1 - \beta T + p^3 T^2)$$

$H^{2,1} \oplus H^{1,2}$ $\qquad$ $H^{3,0} \oplus H^{0,3}$

$\boxed{\text{factors over } \mathbb{Z}\ \forall p}$

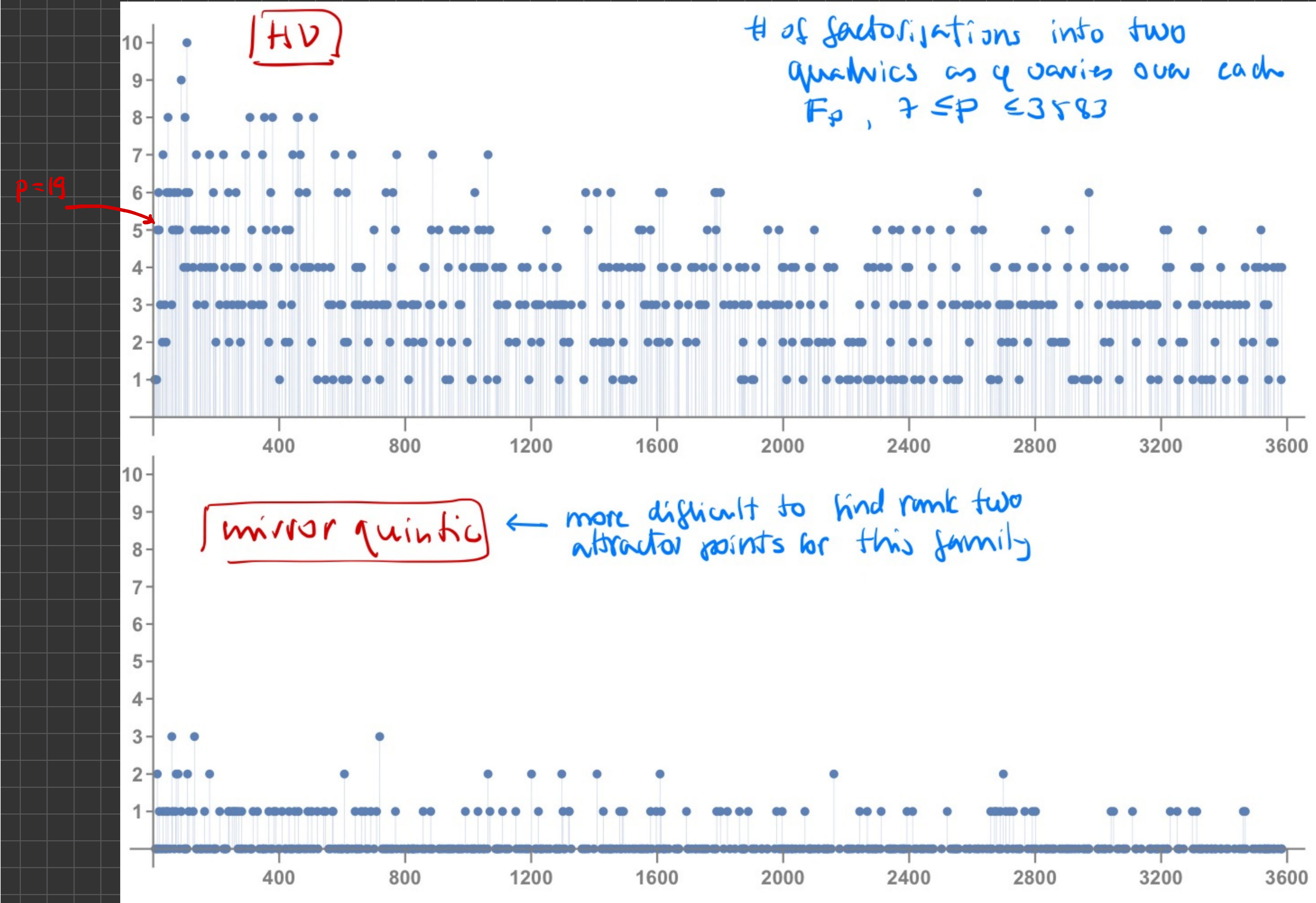
Moreover: expect α, β to be coeffs of modular forms
(Tate & Serre conjectures)

Arithmetic strategy:

make tables of $R_3(T,\varphi)$ for many p & φ
and look for persistent factorisations of R_3
into two quadrics

factorisations occuring when
φ_* is a root of a polynomial
with integer coeffs

[P. Candelas, XD, A Thorne, D v Straten
P. Candelas, XD, D v Straten 04/2021]


HV
# of factorisations into two quadrics as q varies over each $\mathbb{F}_p$, $7 \le p \le 3583$
p=19
mirror quintic
more difficult to find rank two attractor points for this family

We find that for the HV manifold there is always a factorisation when (P Candelas, XD, M Elmi & D v Straten)

- $7\varphi+1=0$

and

- $\varphi^2-66\varphi+1=0$: $\varphi_\pm=33\pm8\sqrt{17}$

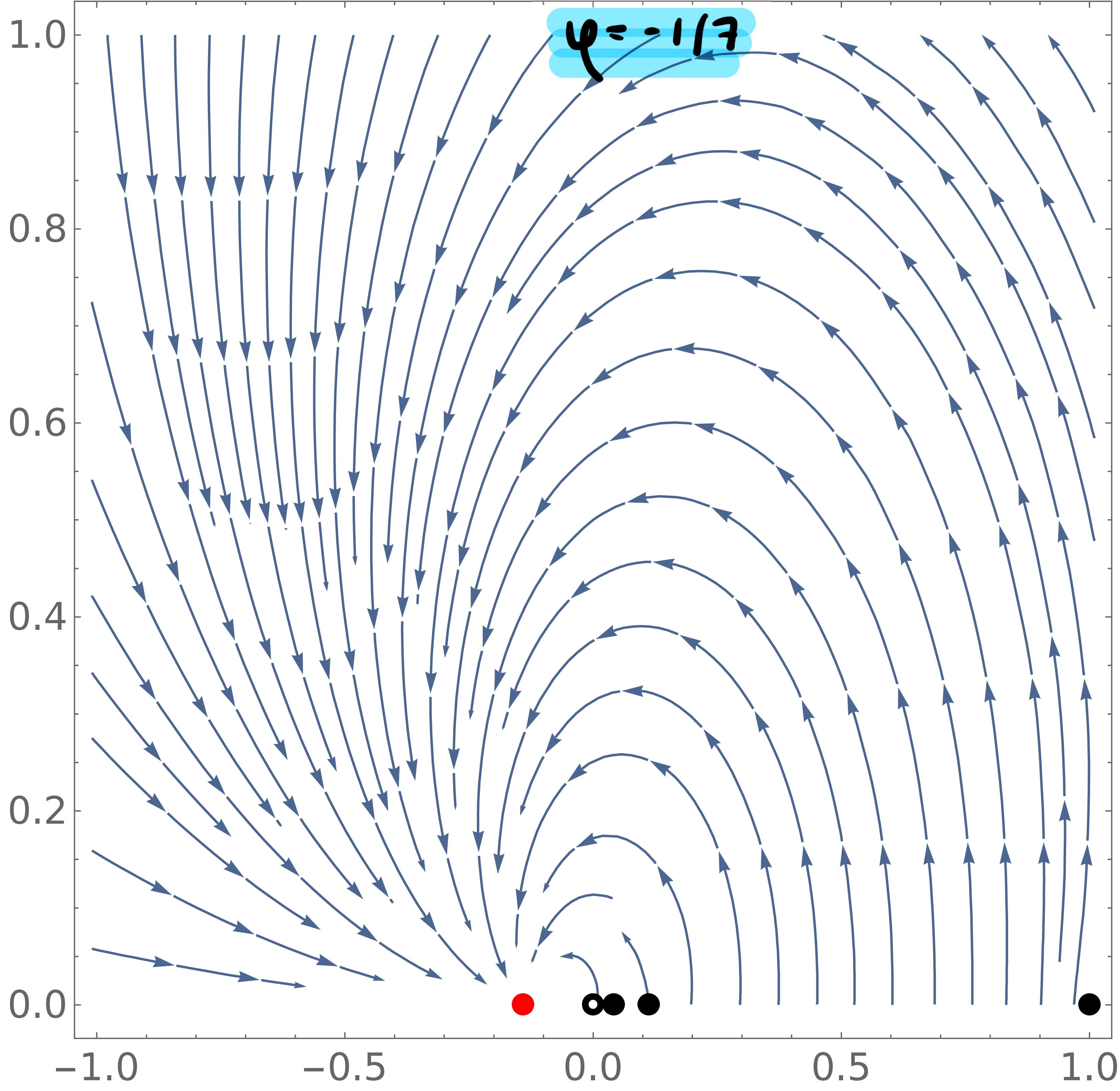
For $p=19$ (say)

$\varphi=-\frac{1}{7}\equiv 8$, $\varphi_\pm\equiv 4, 5$

$(17\equiv 6^2)$

	$p = 19$			
	φ	smooth/sing.	singularity	$R(T)$
conifold	1	singular	1	$(1 - pT)(1 - 20T + p^3T^2)$
	2	smooth		$1 + 4pT + 2pT^2 + 4p^4T^3 + p^6T^4$
	3	smooth		$1 - 8T + 242pT^2 - 8p^3T^3 + p^6T^4$
φ_+	4	smooth		$(1 + 4pT + p^3T^2)(1 - 60T + p^3T^2)$
φ_-	5	smooth		$(1 + 4pT + p^3T^2)(1 - 60T + p^3T^2)$
	6	smooth		$1 + 8T - 318pT^2 + 8p^3T^3 + p^6T^4$
	7	smooth		$1 - 44T - 238pT^2 - 44p^3T^3 + p^6T^4$
$\varphi = -1/7$	8	smooth		$(1 - 2pT + p^3T^2)(1 - 80T + p^3T^2)$
	9	smooth		$(1 + 4pT + p^3T^2)(1 - 160T + p^3T^2)$
	10	smooth		$1 + 12T + 562pT^2 + 12p^3T^3 + p^6T^4$
	11	smooth		$(1 + 4pT + p^3T^2)(1 - 140T + p^3T^2)$
	12	smooth		$1 + 12T + 82pT^2 + 12p^3T^3 + p^6T^4$
	13	smooth		$1 + 178T + 1082pT^2 + 178p^3T^3 + p^6T^4$
	14	smooth		$1 + 12T - 158pT^2 + 12p^3T^3 + p^6T^4$
	15	smooth		$1 + 42T - 2p^2T^2 + 42p^3T^3 + p^6T^4$
conifold	16	singular	$\frac{1}{25}$	$(1 - pT)(1 + 76T + p^3T^2)$
conifold	17	singular	$\frac{1}{9}$	$(1 - pT)(1 - 20T + p^3T^2)$
	18	smooth		$1 - 54T + 322pT^2 - 54p^3T^3 + p^6T^4$

Table 1: *The R-factors for $\varphi \in \mathbb{F}_{19}$. Note the factorisations into two quadrics for the five values $\varphi = 4, 5, 8, 9, 11$.*



φ = -1/7
1.0
0.8
0.6
0.4
0.2
0.0
−1.0
−0.5
0.0
0.5
1.0

There is more information in the tables

there are modular forms

$$R_3(T) = (1 - p\,\alpha_p T + p^3 T^2)(1 - \beta_p T + p^3 T^2)$$

Serre's conjecture (generalizing Taniyama - Weil)

"motives" of length two are modular

algebraically defined part of cohomology

[proof: Dieulefait, Khare & Wintenberger, Kisin]

For $\varphi = -1/7$

α_p & β_p are Fourier coefficients of a modular form for $\Gamma_0(14)$

LMFDB

$f_2 = \sum \alpha_n q^n$ weight 2 14.2.a.a

$f_4 = \sum \beta_n q^n$ weight 4 14.4.a.a

Similarly, for $\varphi_{\pm}$

α_p & β_p $\longrightarrow$ coeffs of modular forms for $\Gamma_1(34) \subset \Gamma_0(34)$

$$SL(2,\mathbb{Z}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod 34$$

$f_2 \longrightarrow$ 34.2.b.a

$f_4 \longrightarrow$ 34.4.b.a

Area of the horizon of the BH

$\varphi = -1/7$

Charges $Q_{kl} = k(4K, -15K, -5, 0) + l(0, 0, 2, 1)$ $K = 1, 2$

(two parameter family of BHs)

let $V_* = \dfrac{7}{\pi} \dfrac{L_4(2)}{L_4(1)}$ $L_4 \to$ L-function associated to f_4

Then

$$\boxed{A(\varphi_*) = 14\pi \left\{ k^2 V_* + \left(l - \frac{5k}{2}\right)^2 \frac{1}{V_*} \right\}} \propto \text{BH entropy}$$

What is the meaning of this?

Outlook

- What makes a CY variety an attractor variety? Why is the HV example so special?

  There must be a geometric reason for the splitting of the Hodge (Hodge conjecture)

- Modularity of CY varieties?

- Mirror symmetry: $L(HV) \longleftrightarrow L(\text{mirror of } HV)$

eg mirror map: $t_* = t(-\frac{1}{7}) = \frac{1}{12} + \frac{5\pi i}{28} \frac{L_4(1)}{L_4(2)}$

...


¡THANKS!