Irreducible Specht modules for symmetric groups and beyond

Louise Sutton

Women at the Intersection of Mathematics and Theoretical Physics Meet in Okinawa

OIST

The symmetric group

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subject to the relations

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Fields of characteristic zero: e.g. $\mathbb{Q}$, $\mathbb{C}$.

Fields of prime characteristic: e.g. finite fields $\mathbb{Z}/p\mathbb{Z}$, p a prime.

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Open Problem

Can we explicitly describe the irreducible modules of $\mathfrak{S}_n$?
What are their dimensions? What are their bases?

Introducing combinatorics

A **partition** λ of n is a non-increasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_i \geqslant \lambda_{i+1}$ for all $i \geqslant 1$ and $\sum_{i=1}^{\infty} \mid \lambda_i \mid = n$.

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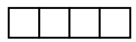

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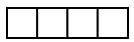
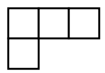
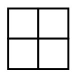
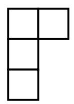
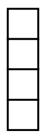
(4)		
$(3,1)$		
(2^2)		
$(2,1^2)$		
(1^4)		

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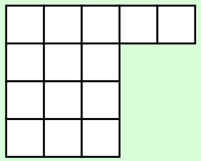
Let $p = 3$. $[(5, 3^3)] =$  is neither 3-regular nor 3-restricted

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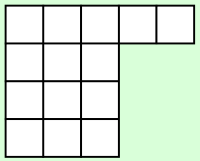
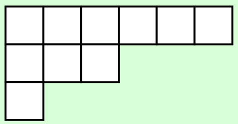
Let $p = 3$. $[(5, 3^3)] =$  is neither 3-regular nor 3-restricted,

but $[(6, 3, 1)] =$  is 3-regular but not 3-restricted.

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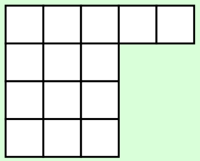
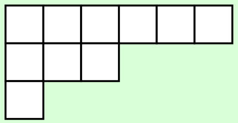
Let $p = 3$. $[(5,3^3)] =$  is neither 3-regular nor 3-restricted,

but $[(6,3,1)] =$  is 3-regular but not 3-restricted.

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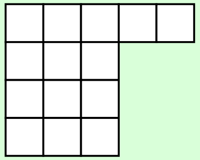
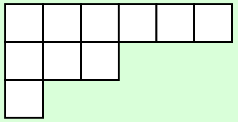
Let $p = 3$. $[(5, 3^3)] =$  is neither 3-regular nor 3-restricted,

but $[(6, 3, 1)] =$  is 3-regular but not 3-restricted.

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What are the dimensions and bases of irreducible $\mathfrak{S}_n$-modules?

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- LEVEL $\ell = 1$: the Iwahori–Hecke algebra $\mathscr{H}(\mathfrak{S}_n)$ of type A.

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The **Hecke algebra of the symmetric group**, $\mathscr{H}(\mathfrak{S}_n)$, is the unital, associative $\mathbb{F}$-algebra with generating set

$$\{T_1, T_2, \ldots, T_{n-1}\}$$

subject to the relations

$$(T_i - q)(T_i + 1) = 0 \qquad \text{for } 1 \leqslant i < n;$$
$$T_i T_j = T_j T_i \qquad \text{for } |i - j| > 1;$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad \text{for } 1 \leqslant i \leqslant n - 2.$$

There is a **cyclotomic Hecke algebra**, $\mathscr{H}_n$, associated to each complex reflection group of type $G(\ell, 1, n) = (\mathbb{Z}/\ell\mathbb{Z}) \wr \mathfrak{S}_n$.

- Level $\ell = 1$: the Iwahori–Hecke algebra $\mathscr{H}(\mathfrak{S}_n)$ of type A.
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A long-standing conjecture

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Aim

First step: *classify the irreducible Specht modules in positive characteristic.*

Hook lengths

Let λ be a partition, and $[\lambda]$ be its Young diagram.

The **hook length** of a box $(a,b) \in [\lambda]$ is

$$h_{ab}^{\lambda} := (\lambda_a - b) + (\lambda'_b - a) + 1$$
$$= \text{arm length} + \text{leg length} + \text{node } (a,b)$$

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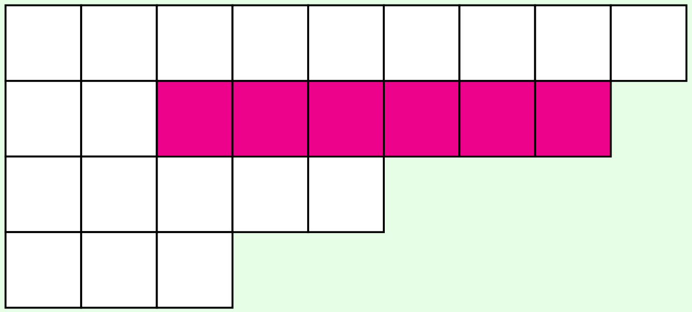

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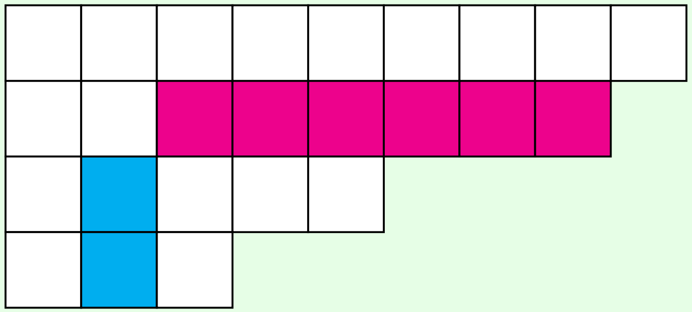
$h_{ab}^{\lambda} =$  $= 6 + 2$

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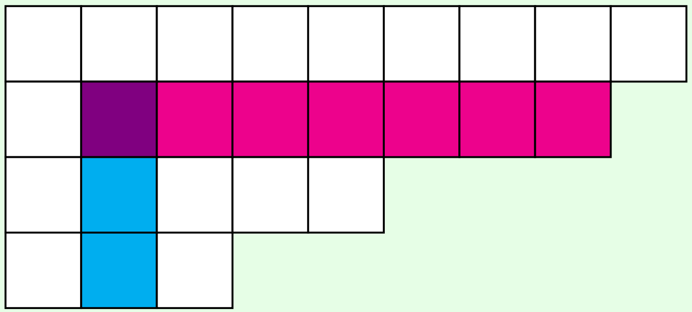

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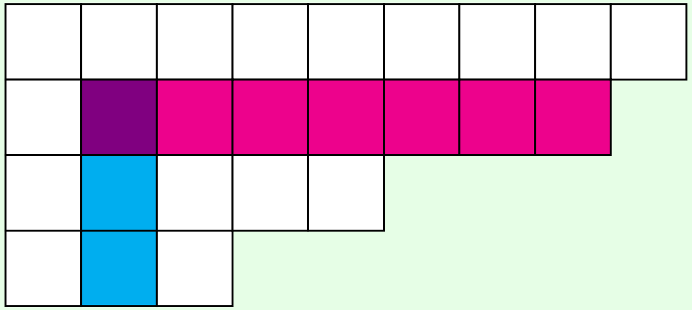
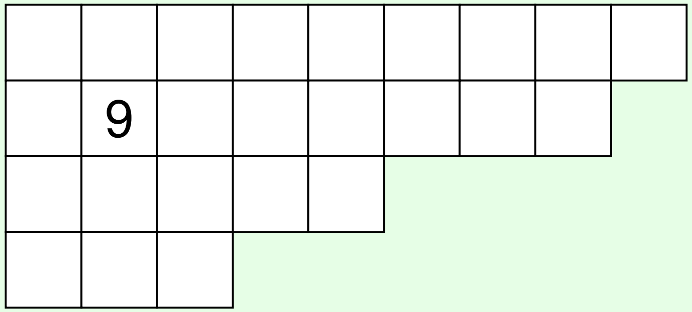
$h_{ab}^{\lambda} =$  $= 6 + 2 + 1 = 9.$

The **hook lengths** of λ are 

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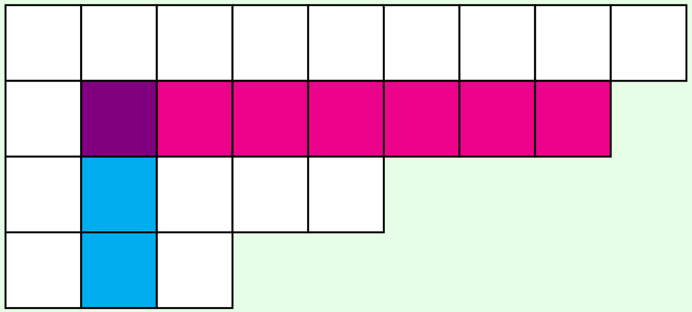
$h^{\lambda}_{ab} =$  $= 6 + 2 + 1 = 9.$

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								1
	9						1	
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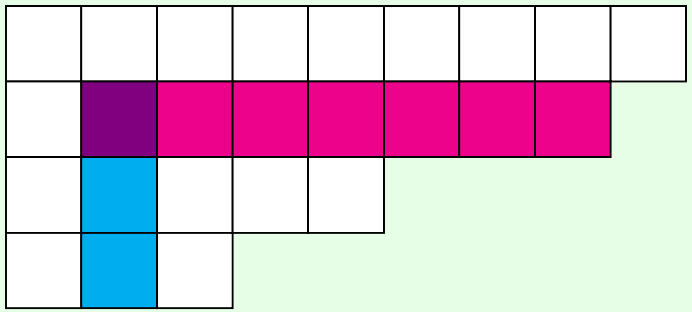
$h^{\lambda}_{ab} =$  $= 6 + 2 + 1 = 9.$

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	9					2	1	
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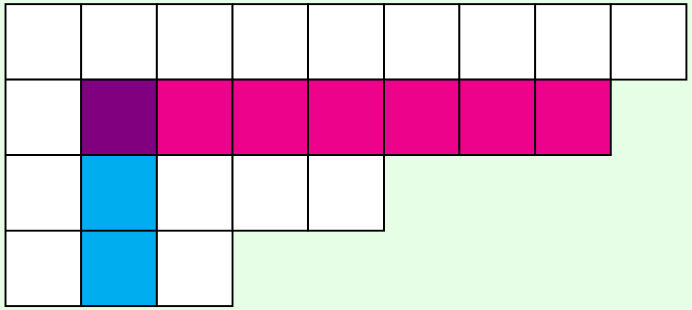
$h_{ab}^{\lambda} =$  $= 6 + 2 + 1 = 9.$

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							3	1
	9				3	2	1	
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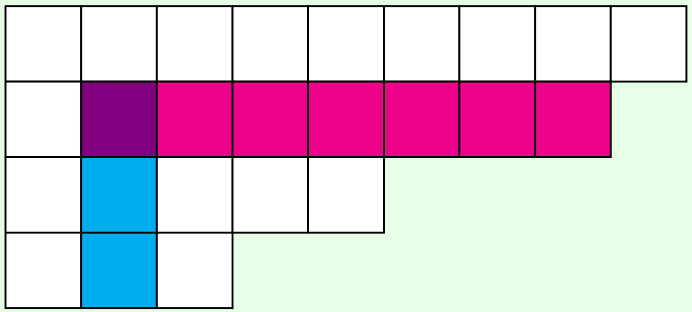
$h^{\lambda}_{ab} =$  $= 6 + 2 + 1 = 9.$

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						4	3	1
	9				3	2	1	
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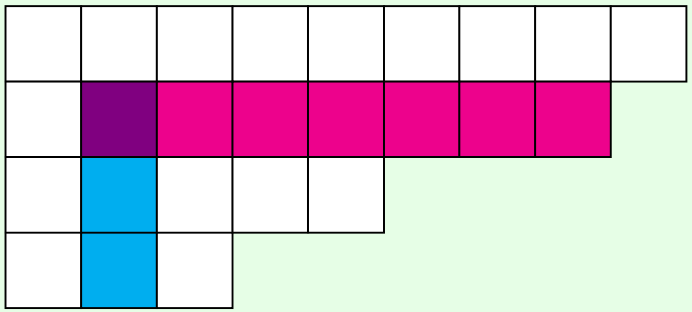
$h_{ab}^{\lambda} =$  $= 6 + 2 + 1 = 9.$

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					5	4	3	1
	9			5	3	2	1	
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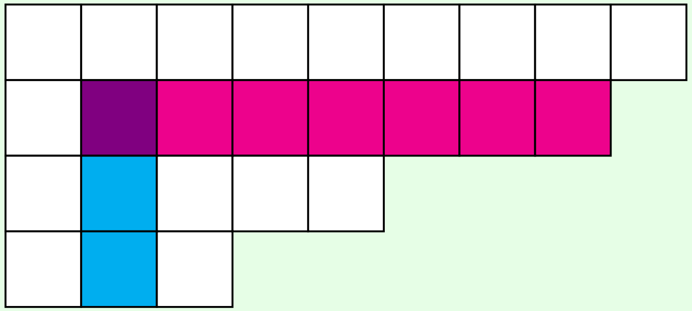
$h^{\lambda}_{ab} =$  $= 6 + 2 + 1 = 9.$

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12	11	10	8	7	5	4	3	1
10	9	8	6	5	3	2	1	
6	5	4	2	1				
3	2	1						

IRREDUCIBLE SPECHT MODULES FOR $\mathscr{H}(\mathfrak{S}_n)$

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PARTITIONS INDEXING IRREDUCIBLE SPECHT MODULES

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17	14	10	8	7	5	4	3	2	1
11	8	4	2	1					
8	5	1							
6	3								
5	2								
4	1								
2									
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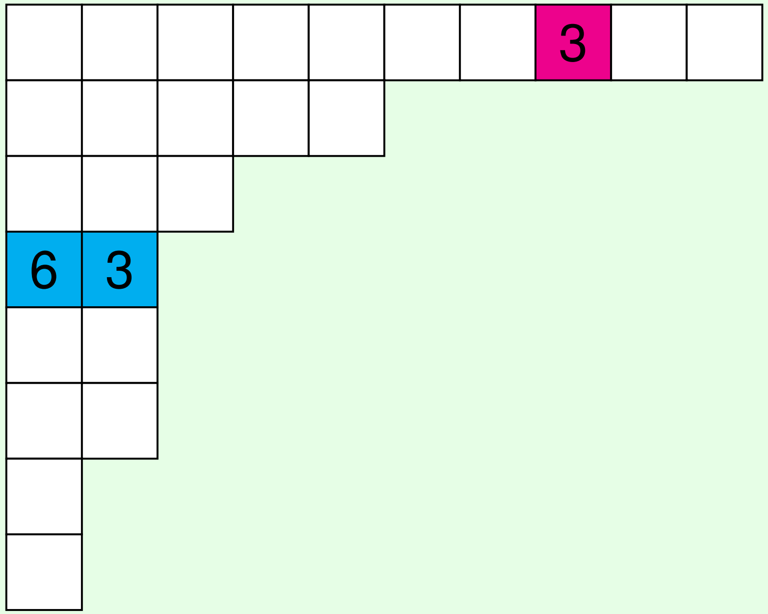
$[(10, 5, 3, 2^3, 1^2)] =$ 

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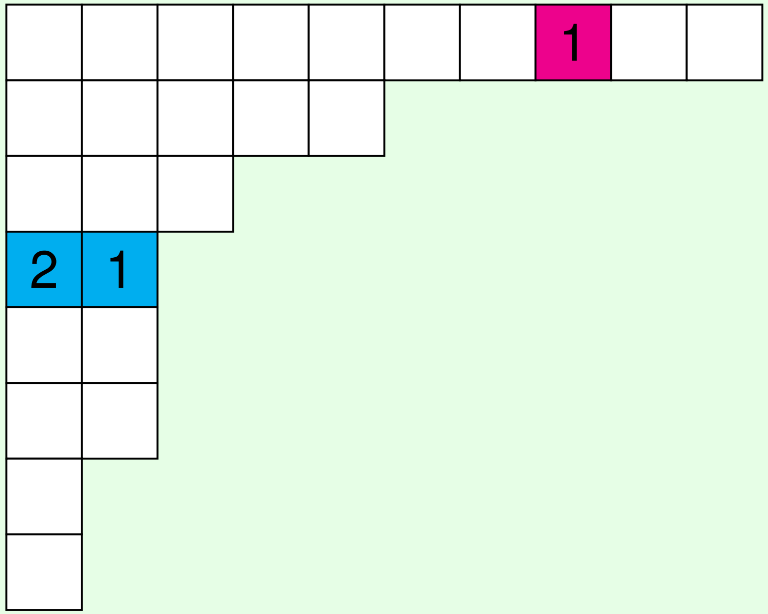

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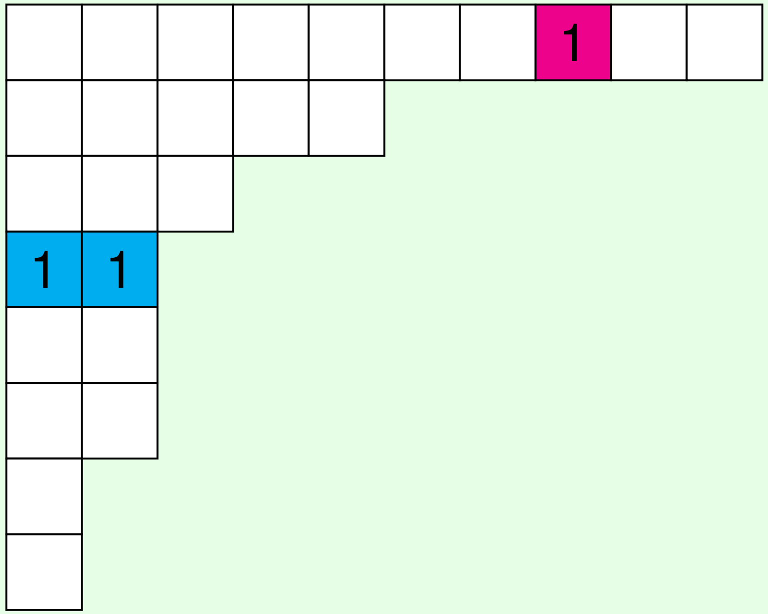

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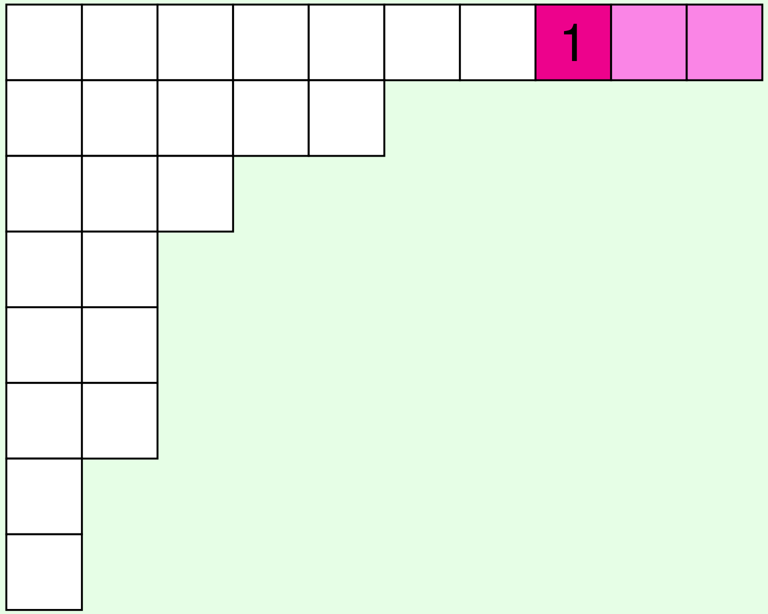

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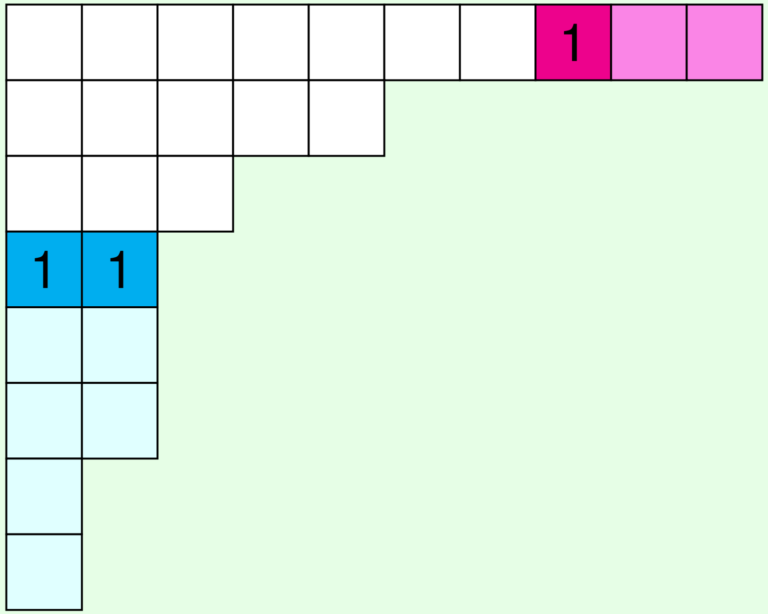

- (3) is 3-regular;
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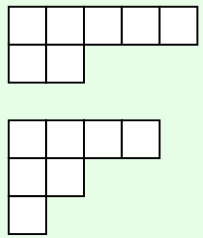
$((5,2),(4,2,1))$ is a bipartition of 14, with Young diagram .

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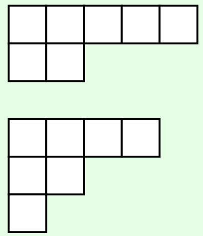
$((5,2),(4,2,1))$ is a bipartition of 14, with Young diagram .

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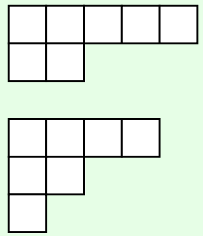
$((5,2),(4,2,1))$ is a bipartition of 14, with Young diagram .

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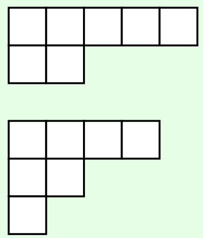
$((5,2),(4,2,1))$ is a bipartition of 14, with Young diagram .

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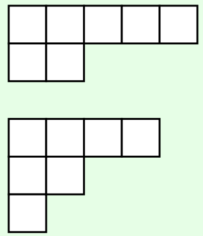
$((5,2),(4,2,1))$ is a bipartition of 14, with Young diagram .

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Can we classify irreducible Specht modules $S(\lambda, \mu)$?

Conjectural classification

Conjecture (Joint work with Matthew Fayers)

The "unrestrictable" Specht module $S(\lambda, \mu)$ *is* IRREDUCIBLE *if and only if*

CONJECTURAL CLASSIFICATION

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Conjectural classification

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