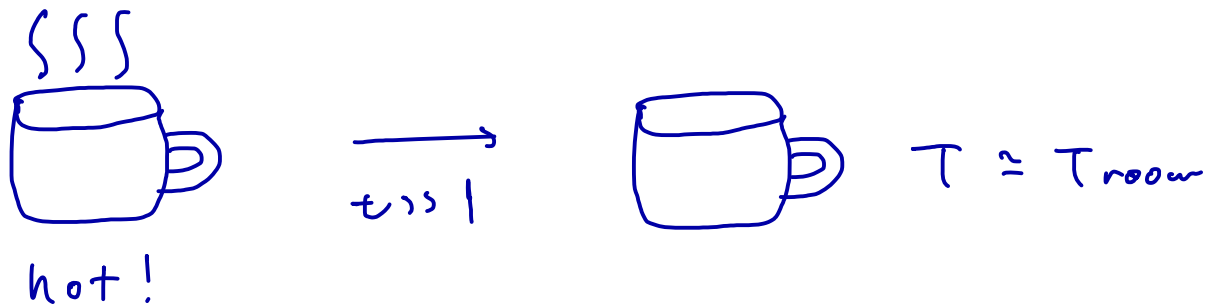
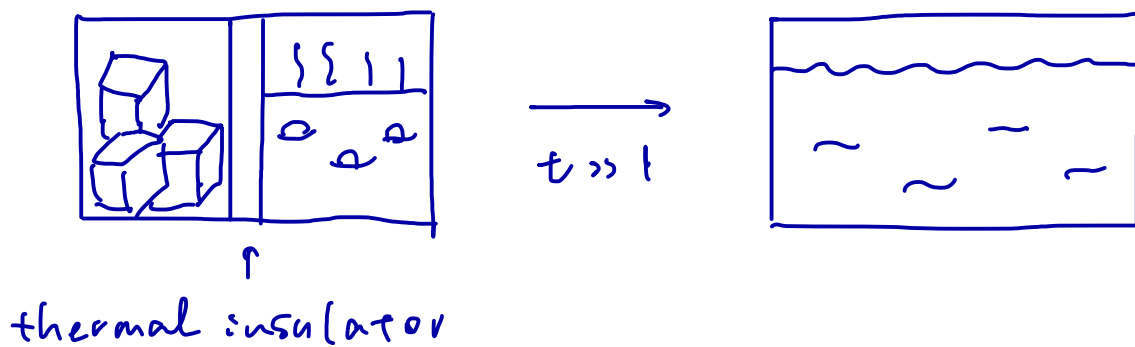


# Thermalization & Relaxation of isolated quantum systems

"Thermalization"



Hot coffee cools down due to heat exchange with external system.



An isolated system is also thermalized.

- How about quantum systems?
- If not thermalized, why?
- Microscopic description of thermalization?

We use the knowledge of

- Quantum mechanics

← quantum systems

- Statistical mechanics

← in order to understand the mechanism of thermalization

( connecting micro world to macro world )

- Integrable systems

← representative examples in which thermalization does not occur.

[ CM arXiv.2002.01069 ]

Quantum  
mechanics



Statistical  
mechanics



◦ Relaxation of quantum integrable systems

# Outline

§ 1. Postulates of quantum mechanics

§ 2. Postulates of statistical mechanics

§ 3. Thermalization vs. Integrable systems

§ 4. Summary

# § 1. Postulates of quantum mechanics

Quantum systems are mathematically formulated by using "linear algebra".

## Notations

$\mathcal{H}$ : Hilbert space  $\leftarrow$  where  $q$ -systems live on.

$z^*$ : complex conjugate of the complex number  $z$

$|\psi\rangle$ : vector (called as "ket").

$\langle\psi|$ : vector dual to  $|\psi\rangle$  (called as "bra").

$\langle\phi|\psi\rangle$ : inner product between the vectors  $|\phi\rangle$  and  $|\psi\rangle$ .

$|\phi\rangle \otimes |\psi\rangle$ : tensor product of  $|\phi\rangle$  and  $|\psi\rangle$ .

"  
 $|\phi\rangle|\psi\rangle$

$A^*$ : complex conjugate of the matrix  $A$ .

$A^\dagger$ : Hermitian conjugate or adjoint of the matrix  $A$ .

"  
 $(A^\dagger)^*$

$\langle\phi|A|\psi\rangle$ : inner product between  $|\phi\rangle$  and  $A|\psi\rangle$

equivalently, inner product between  $A^\dagger|\phi\rangle$  and  $|\psi\rangle$ .

## Hilbert space (finite-dim. case)

Complex vector space equipped with inner products.

### Inner products

$$[ \ , \ ] = \langle \ | \ \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C} \text{ s.t.}$$

for  $|v\rangle, |w\rangle \in \mathcal{H}$ ,  $\lambda_j \in \mathbb{C}$ ,

(i) Bilinear

$$\langle v | \left( \sum_j \lambda_j |w_j\rangle \right) = \sum_j \lambda_j \langle v | w_j \rangle$$

$$\left( \sum_j \lambda_j^* \langle v_j | \right) |w\rangle = \sum_j \lambda_j^* \langle v_j | w \rangle$$

(ii) Hermitian

$$\langle v | w \rangle = \langle w | v \rangle^*$$

(iii) Positive definite

$$\langle v | v \rangle \geq 0 \quad (= 0 \text{ iff } |v\rangle = 0).$$

ex: A closed bracket produces a scalar.



# Tensor products

## • Tensor product of vectors

$\mathcal{H}_1, \mathcal{H}_2$ : Hilbert space

$$\left( \begin{array}{l} \cdot \dim \mathcal{H}_1 = m \\ \cdot \dim \mathcal{H}_2 = n \end{array} \right.$$

tensor product

$\mathcal{H}_1 \otimes \mathcal{H}_2$ :  $mn$ -dimensional vector space.

$$|v\rangle \otimes |w\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2.$$

$$|v\rangle |w\rangle$$

$$|v, w\rangle.$$

Especially,

$$\left( \begin{array}{l} \langle i | \rangle_{i=1, \dots, m} : \text{orthonormal basis for } \mathcal{H}_1 \\ \langle j | \rangle_{j=1, \dots, n} : \text{orthonormal basis for } \mathcal{H}_2 \end{array} \right.$$

$$\Rightarrow \langle i | \otimes \langle j | \rangle_{\substack{i=1, \dots, m \\ j=1, \dots, n}} : \text{basis for } \mathcal{H}_1 \otimes \mathcal{H}_2.$$

## Properties

(i)  $\forall z \in \mathbb{C}, |v\rangle \in \mathcal{H}_1, |w\rangle \in \mathcal{H}_2,$

$$\begin{aligned} z(|v\rangle \otimes |w\rangle) &= (z|v\rangle) \otimes |w\rangle \\ &= |v\rangle \otimes (z|w\rangle) \end{aligned}$$

(ii)  $\forall |v_1\rangle, |v_2\rangle \in \mathcal{H}_1, |w\rangle \in \mathcal{H}_2,$

$$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$$

(iii)  $\forall |v\rangle \in \mathcal{H}_1, |w_1\rangle, |w_2\rangle \in \mathcal{H}_2,$

$$|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$$

• Tensor product of operators

$$\begin{pmatrix} A : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \\ B : \mathcal{H}_2 \rightarrow \mathcal{H}_2 \end{pmatrix}$$

$$\rightarrow \underbrace{(A \otimes B)}_{\substack{\mathcal{H}_1 \otimes \mathcal{H}_2 \\ \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2}} \left( \underbrace{|v\rangle}_{\mathcal{H}_1} \otimes \underbrace{|w\rangle}_{\mathcal{H}_2} \right) := A|v\rangle \otimes B|w\rangle.$$

Inner product on  $\mathcal{H}_1 \otimes \mathcal{H}_2$

$$\left( \sum_i \overset{\mathcal{H}_1}{a_i} \overset{\mathcal{H}_2}{|v_i\rangle} \otimes \overset{\mathcal{H}_2}{|w_i\rangle}, \sum_j \overset{\mathcal{H}_1}{b_j} \overset{\mathcal{H}_1}{|v_j'\rangle} \otimes \overset{\mathcal{H}_2}{|w_j'\rangle} \right)$$

$$:= \sum_{i,j} a_i^* b_j \langle v_i | v_j' \rangle \langle w_i | w_j' \rangle.$$

$\Rightarrow$ 
 $\left. \begin{array}{l} \text{Bilinearity} \\ \text{Hermiticity} \\ \text{Positive definiteness} \end{array} \right\} \text{ satisfied.}$

# Matrix representations (Kronecker product)

For  $A \in M(m, n; \mathbb{C})$  and  $B \in M(p, q; \mathbb{C})$ ,

$$A \otimes B = \overbrace{\begin{pmatrix} A_{11} \underbrace{B}_{n \times q} & \cdots & A_{1n} \underbrace{B}_{n \times q} \\ \vdots & & \vdots \\ A_{m1} \underbrace{B}_{n \times q} & \cdots & A_{mn} \underbrace{B}_{n \times q} \end{pmatrix}}^{nq} \left. \vphantom{\begin{pmatrix} A_{11} B & \cdots & A_{1n} B \\ \vdots & & \vdots \\ A_{m1} B & \cdots & A_{mn} B \end{pmatrix}} \right\} mp$$

matrix

Ex.)

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ \vdots \\ v_2 w_1 \\ v_2 w_2 \end{pmatrix}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ \vdots & \vdots & \vdots & \vdots \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}$$

## Operator functions

$$f: \mathbb{C} \rightarrow \mathbb{C} \text{ (normal function)}$$

"Matrix function"  $\rightarrow$  uniquely defined

$$f(A) := \sum_a f(a) |a\rangle\langle a|$$

for a normal matrix  $A = \sum_a |a\rangle\langle a|$ .

spectral decomposition

## Commutator and anti-commutator

- Commutator

$$[A, B] := AB - BA$$

$A$  &  $B$  are simultaneously diagonalizable if Hermitian.

- Anti-commutator

$$\{A, B\} := AB + BA$$

## Postulate 1 ) State space

Isolated physical system  $\Leftrightarrow$  Hilbert space  $\mathcal{H}$

Quantum state  $\Leftrightarrow$  Unit vector  $|\psi\rangle$   
"state vector"

### Remark )

Globally phase shifted states  $e^{i\theta} |\psi\rangle$  are identified with  $|\psi\rangle$ .

## Postulate 2 ) Physical quantities ( Observables )

Physical quantity  $\Leftrightarrow$  Self-adjoint operator  
( Hermitian )  
in  $\mathcal{H}$ .

Postulate 3) Born's probability rule.

$A : \mathbb{C}^N \rightarrow \mathbb{C}^N$  Hermite  
Observable  
Physical quantity

$$\Rightarrow A = \sum_{n=1}^N \alpha_n |\alpha_n\rangle \langle \alpha_n| \quad : \text{spectral decomp.}$$

$\uparrow$  eigenvalue     $\uparrow$  eigenvector

$$= \sum_{n=1}^N \alpha_n P(\alpha_n)$$

$\uparrow$  projector onto  $\{|\alpha_n\rangle\}$

In quantum measurement on  $|\psi\rangle \in \mathcal{H}$ ,

• Eigenvalues of  $A \Leftrightarrow$  Measurement outcomes

• Probability to obtain the outcome "a"

$$: p(\alpha) = \langle \psi | P(\alpha)^\dagger P(\alpha) | \psi \rangle$$

$$= \| P(\alpha) | \psi \rangle \|^2$$

$$= \underbrace{|\langle \alpha | \psi \rangle|^2}_{\text{square of the amplitude}}$$

$$\geq 0 \quad (\text{non-negative}).$$

$$\sum_{\alpha} |\langle \alpha | \psi \rangle|^2 = \sum_{\alpha} \langle \psi | \alpha \rangle \langle \alpha | \psi \rangle$$

$$= \langle \psi | \psi \rangle$$

$$= 1 \quad \text{Sum rule.}$$

Expectation value of  $A$  on  $|\psi\rangle$ .

---

$$\begin{aligned}\langle A \rangle &= \sum_{n=1}^N \alpha_n P(\alpha_n) \\ &= \sum_{n=1}^N \alpha_n \underbrace{|\langle \alpha_n | \psi \rangle|^2}_{\langle \psi | \alpha_n \rangle \langle \alpha_n | \psi \rangle} \\ &= \langle \psi | \underbrace{\sum_{n=1}^N \alpha_n |\alpha_n\rangle \langle \alpha_n|}_{A} | \psi \rangle \\ &= \underline{\langle \psi | A | \psi \rangle}.\end{aligned}$$

Postulate 4) (Time) Evolution

Closed quantum system

$$\begin{array}{ccc} \boxed{|\psi\rangle} & \longrightarrow & \boxed{|\psi'\rangle} = \underline{U(t_1, t_2)} |\psi\rangle \\ t = t_1 & & t_2 \end{array}$$

unitary operator  
 $\approx e^{-iH(t_2 - t_1)}$

$$\Leftrightarrow i \frac{d}{dt} |\psi\rangle = \underline{H} |\psi\rangle \quad ; \text{ Schrödinger eq.}$$

Hermitian operator

(Hamiltonian i energy operator)

# Density operator

- The system is in one of the state out of  $\{|\psi_i\rangle\}$ .
- We only know the prob. for the system in  $|\psi_i\rangle$  is  $p_i$ .

$$\boxed{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad \sum_i p_i = 1$$

↑ ↑

density matrix       $\{p_i, |\psi_i\rangle\}$  : ensemble of pure states.

## Pure state and mixed state

$$\boxed{|\psi\rangle} = \boxed{\rho = |\psi\rangle\langle\psi|} \quad : \text{pure state}$$

( $\text{tr } \rho^2 = 1$ )

$$\boxed{\begin{array}{l} |\psi_i\rangle \\ \text{w.p. } p_i \end{array}} = \boxed{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|}$$

dist.  $\{p_i\}$       mixture of pure state

( $\text{tr } \rho^2 < 1$ )

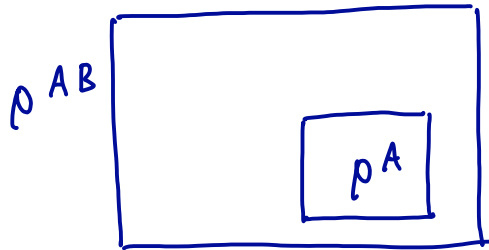
# Properties of density operator

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad : \text{density operator}$$

- $$\Leftrightarrow \left( \begin{array}{l} \bullet \text{ Trace condition } \text{tr } \rho = 1 \\ \bullet \text{ Positivity condition } \rho \geq 0. \end{array} \right.$$

## Reduced density matrix

↳ Description of the subsystem.



- Density matrix of the whole system  $\rho^{AB}$
- Density matrix of the subsystem

$$\rho^A := \text{tr}_B (\rho^{AB})$$

partial trace

mixed state

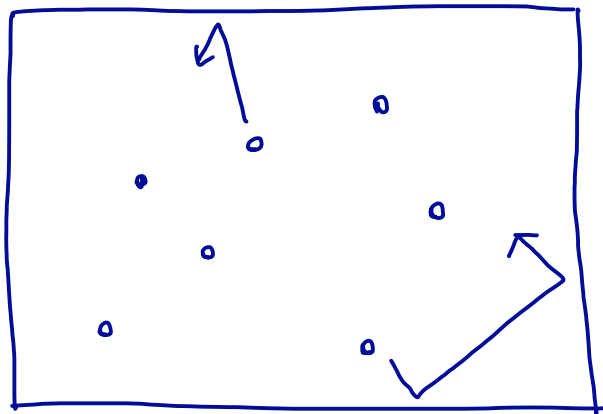
if  $\rho^{AB}$  is an entangled pure state.

$$\begin{aligned} & \text{tr}_B \left( |a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2| \right) \\ & := |a_1\rangle\langle a_2| \text{tr} (|b_1\rangle\langle b_2|) \end{aligned}$$

## § 2. Postulates of statistical mechanics

Aim of statistical mechanics :

to understand macroscopic properties  
from microscopic description.



Classical Newton's eq. of motion

$$\left\{ \begin{array}{l} m\ddot{x}_1 = F(x_1, x_2, \dots) \\ m\ddot{x}_2 = F(x_1, x_2, \dots) \\ \vdots \\ m\ddot{x}_{\text{tot}} = F(x_1, x_2, \dots) \end{array} \right.$$

How do we solve it ?

⇒ Use statistics instead of  
solving Newton eq.

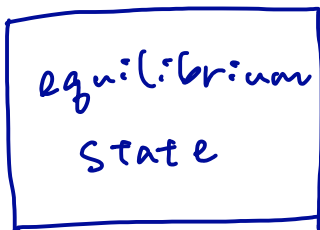
# Equilibrium

Macroscopic thermodynamics



Microscopic mechanics + statistics

Postulate 1 )

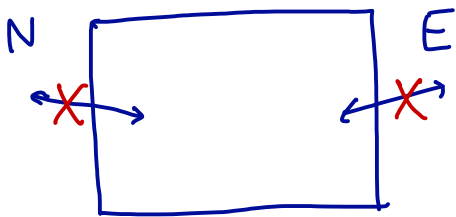


A few thermodynamic var.

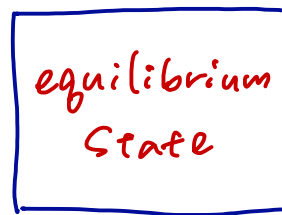
$E, V, N, \dots$

finite # in thermodynamic  
limit.

Postulate 2 )



$t \gg 1$

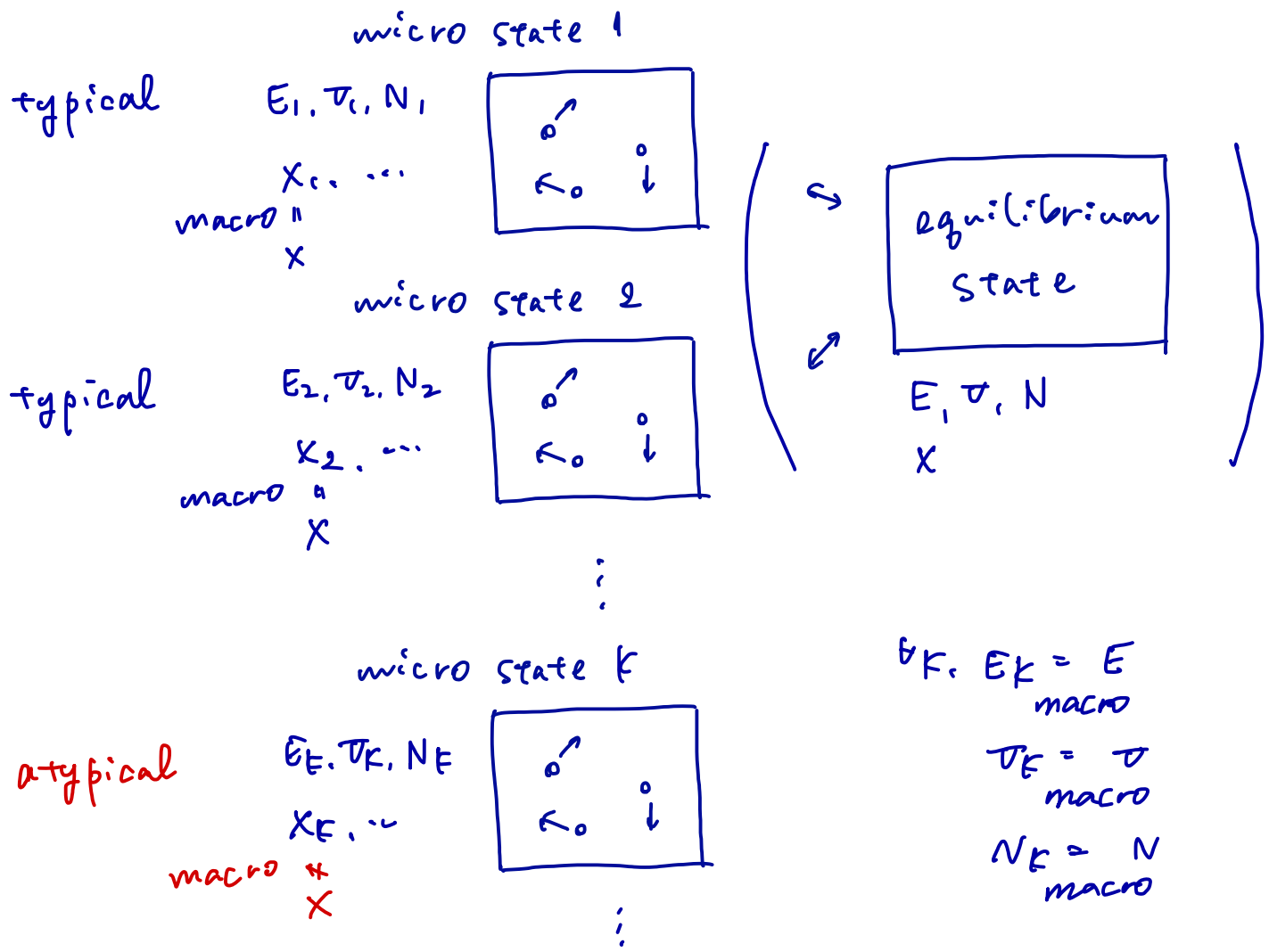


Macroscopic system

- Any macro variable = const.
- Any macro current = 0.

# Postulate 3 )

Almost all microstates are indistinguishable by macro variables.



# Postulate 4 )

Properties of equilibrium state

= Properties of corresponding typical micro states.

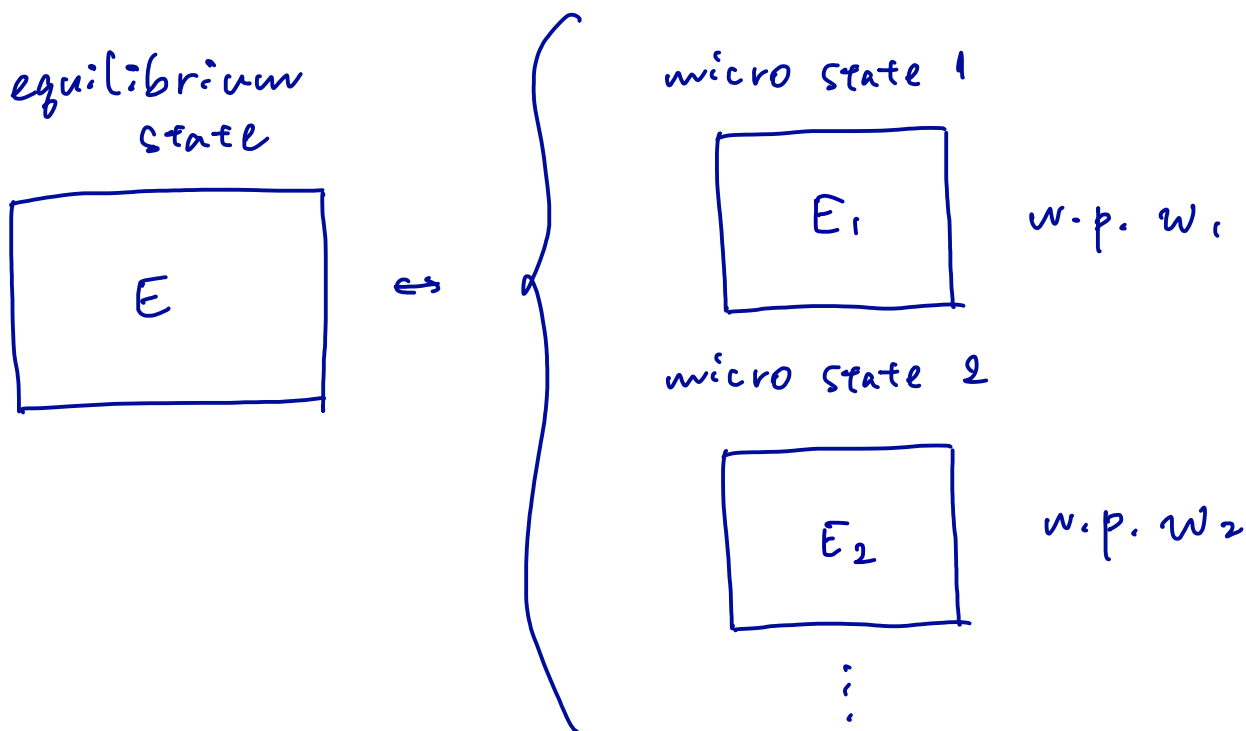
# Microcanonical ens.

Postulate 5) Principle of equal prob.

In an isolated system, microstates with energy  $E \in [E - \delta E, E]$  are realized with equal prob.

$$w_1 = w_2 = \dots = \frac{1}{\underbrace{W(E)}}$$

# of micro states with energy  $\in [E - \delta E, E]$ .



$$w_j = \begin{cases} \frac{1}{W(E)} \\ 0 \end{cases}$$

$$E_j \in [E - \delta E, E]$$

otherwise macroscopically small

( $E_j$  are macroscopically undistinguishable.)

Expectation values of a macro var.  $X$  :

$$\langle X \rangle = \sum_i \frac{w_i}{W(E)} X(x_i, p_i)$$

$\frac{1}{W(E)}$  the value of  $X$  computed in the " $i$ "-th micro state.

Thermodynamics uses a macro var. which is not a function of  $x_i$  and  $p_i$ .

↓

Entropy

Postulate 5) Boltzmann's relation

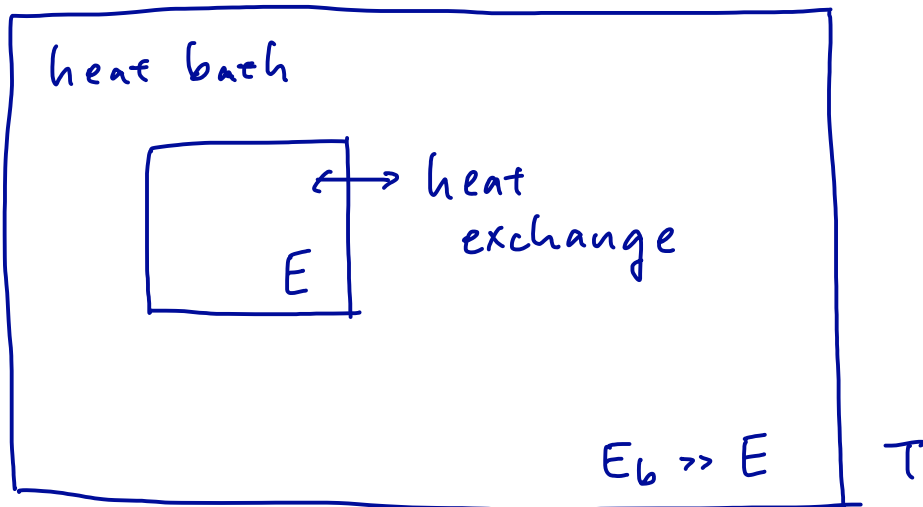
$$S(E) = k_B \ln W(E).$$

$s_i$

$$(1.38 \times 10^{-23} \text{ J/K})$$

( Boltzmann const. )

# Canonical ens.



Q. ) Prob. to obtain the microstate of the subsystem with energy  $E$ ?

Prob. to obtain the subsystem in the microstate  $n$ :

$$p(E_n) = \frac{W_b(E_t - E_n)}{\sum_{E'} W_b(E_t - E') w(E')}$$

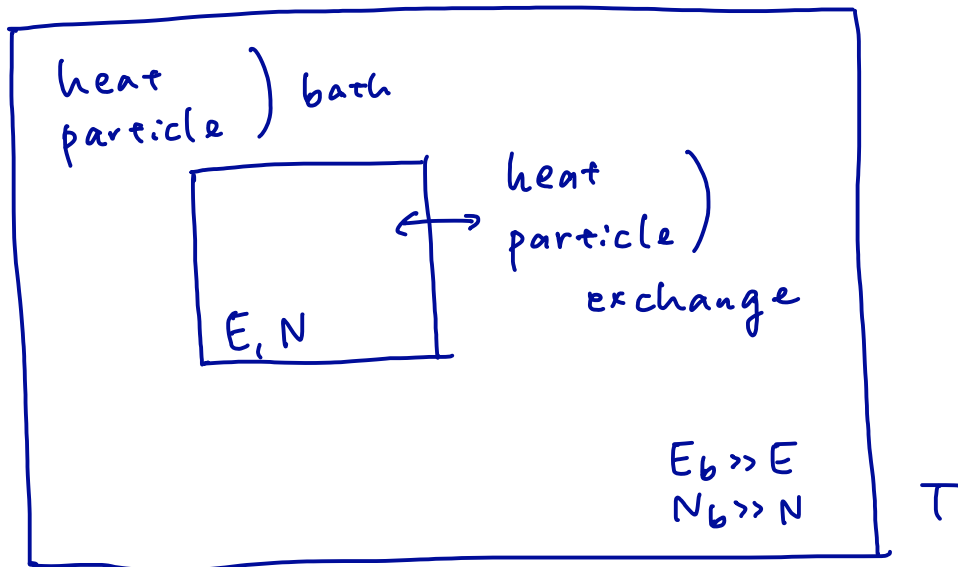
⊙ principle of equal prob.

$$= \frac{1}{Z} e^{-\frac{E_n}{k_B T}} \quad \text{⊙ Boltzmann's relation}$$

canonical (Gibbs) distribution

$$Z = \sum_n e^{-\frac{E_n}{k_B T}} \quad \text{"partition func."}$$

# Grand canonical ensemble



Prob. to obtain the microstate  $n$ :

$$p_n = \frac{1}{\Xi} e^{-\frac{1}{k_B T} (E_n - \mu N_n)} \quad \left( \sum_n p_n = 1 \right).$$

grand canonical dist.

( dist. of subsystem attached to the heat & particle bath with  $E$  &  $N$  ) .

$$\Xi = \sum_n e^{-\frac{1}{k_B T} (E_n - \mu N_n)}$$

# Combining quantum mech. & stat. mech.

## Postulate 1')

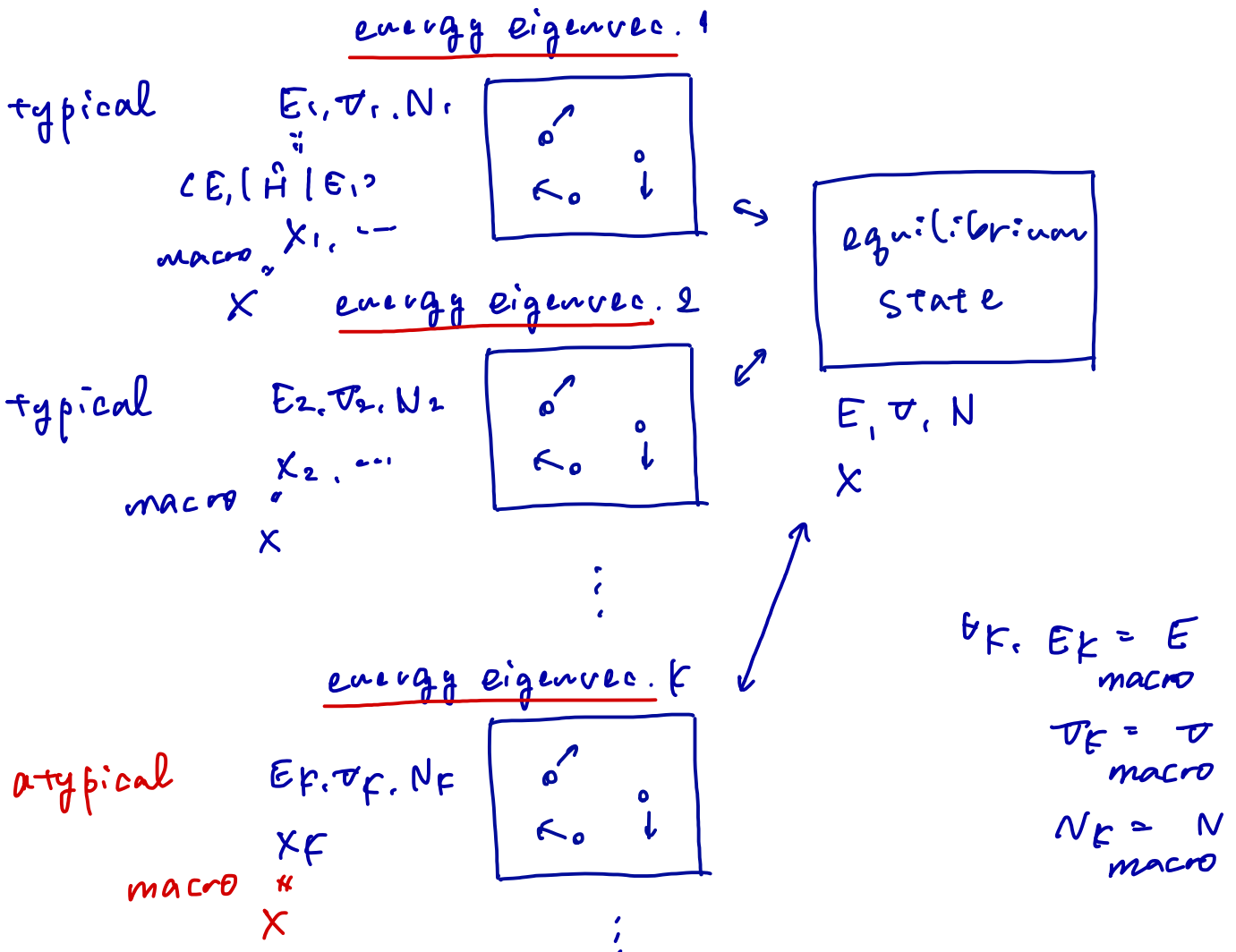
A generic quantum system reaches equilibrium after enough time.

## Postulate 2')

An equilibrium state is characterized by a few thermodynamic operators.

## Postulate 3') : Eigenstate thermalization hypothesis (ETH)

Almost all energy eigenvec. are indistinguishable by macro variables.



Postulate 4')

Properties of equilibrium state

= Properties of corresponding typical energy eigenvec.

Postulate 5')

Expectation values of a macro operator  $X$  :

$$\langle X \rangle = \frac{1}{W(E)} \sum_i \langle E_i | X | E_i \rangle$$

the expectation value of  $X$   
measured on  $|E_i\rangle$ .

$$= \text{tr} \left( \underbrace{\rho_{MC}}_{} X \right)$$
$$\sum_{E_i \in [E-\delta E, E]} \frac{1}{W(E)} |E_i\rangle \langle E_i|.$$

: microcanonical density matrix.

$W(E)$  : # of energy eigenvec. with  
 $E_i \in [E-\delta E, E]$ .

⇒ Principle of equal prob.

cf. )

Density matrix for canonical ens.

$$\rho_{GE} = \frac{1}{Z} e^{-\beta H}$$

$H$  : Hamiltonian

$\beta \in \mathbb{R}$  : inverse temperature

$$Z = \text{tr } \rho_{GE}$$

$$= \sum_n \langle E_n | e^{-\beta H} | E_n \rangle$$

$$= \sum_n e^{-\beta E_n}$$

Density matrix for grandcanonical ens.

$$\rho_{GCE} = \frac{1}{\bar{Z}} e^{-\beta(H - \mu N)}$$

$N$  : particle number

$\mu \in \mathbb{R}$  : chemical potential

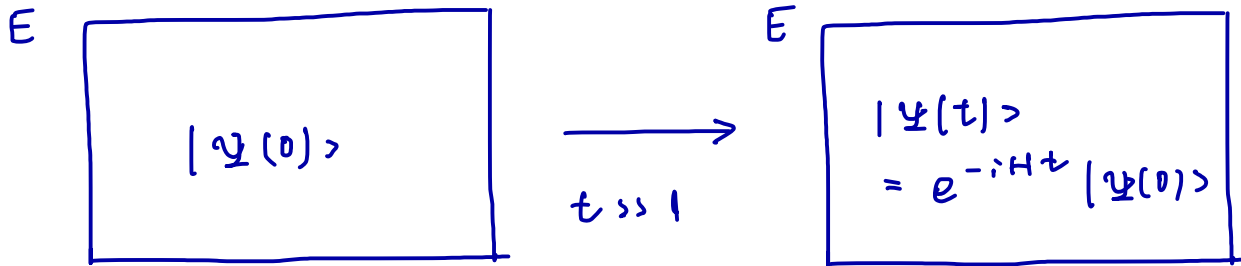
$$\bar{Z} = \text{tr } \rho_{GCE}$$

$$= \sum_n \langle E_n, N_n | e^{-\beta(H - \mu N)} | E_n, N_n \rangle$$

$$= \sum_n e^{-\beta(E_n - \mu N_n)}$$

### § 3. Thermalization vs. integrable systems

#### Thermalization of generic quantum systems



Thought experiment )

$$\text{Let } |\Psi(0)\rangle = \sum_n c_n |E_n\rangle, \quad E_n \in (E - \delta E, E]$$

$$\sum_n |c_n|^2 = 1.$$

$$\begin{aligned} \lim_{t \rightarrow \infty} |\Psi(t)\rangle \langle \Psi(t)| &= \lim_{t \rightarrow \infty} e^{-iHt} |\Psi(0)\rangle \langle \Psi(0)| e^{iHt} \\ &= \lim_{t \rightarrow \infty} \sum_{n,m} e^{-i(E_n - E_m)t} c_n c_m^* |E_n\rangle \langle E_m| \\ &\neq \frac{1}{W(E)} \sum_n |E_n\rangle \langle E_n| \end{aligned}$$

never reaches the MC density matrix.  
mixed state.

\* Do isolated quantum systems thermalize?

## Definitions )

The system shows relaxation if

$$\delta X^2 \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle X \rangle^2(t) dt - \left( \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle X \rangle(t) dt \right)^2$$

$$\ll \|X\|^2$$

operator norm

The system thermalizes if

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle X \rangle(t) dt = \langle X \rangle_{\text{e.f.}}$$

## Remarks )

Relaxation occurs for most initial states of non-resonance systems.

(cf)

$$|\psi(0)\rangle = \sum_n c_n |E_n\rangle, \quad E_n \in [E - \delta E, E]$$

$$\Rightarrow \delta X^2 = \sum_{n \neq m} \sum_{k \neq l} c_n^\dagger c_m c_k c_l e^{i(E_n - E_m - E_k + E_l)t} \times \langle E_n | X | E_m \rangle \langle E_k | X | E_l \rangle^*$$

$$= \sum_{n \neq m} \sum_{k \neq l} c_n^\dagger c_m c_k c_l \delta_{E_n - E_m, E_k - E_l} \times \langle E_n | X | E_m \rangle \langle E_k | X | E_l \rangle^*$$

\* Non-resonance condition

$$E_n - E_m = E_f - E_l \neq 0 \Rightarrow n=f, m=l.$$

$$\Rightarrow \delta X^2 = \sum_{n,m} |c_n|^2 |c_m|^2 |\langle E_n | X | E_m \rangle|^2$$

$$\leq \sum_n |c_n|^4 \langle E_n | X X^\dagger | E_n \rangle$$

$$\leq \underbrace{\sum_n |c_n|^4}_{\text{red bracket}} \|X\|^2$$

$\approx$  # of eigenvectors included in  $|\psi(0)\rangle$ .

Sufficient condition for thermalization:

\* Strong eigenstate thermalization hypothesis

(ETH)

$$\langle \psi(t) | X | \psi(t) \rangle$$

$$= \langle \psi(0) | e^{iHt} X e^{-iHt} | \psi(0) \rangle$$

$$\sum_{E_a \in (E-\delta E, E)} c_a |E_a\rangle, \quad \sum_{E_a \in (E-\delta E, E)} |c_a|^2 = 1$$

$$= \sum_{\substack{E_a, E_b \\ E \in (E-\delta E, E)}} c_a^* c_b e^{i(E_a - E_b)t} \langle E_a | X | E_b \rangle$$

$$= \sum_{E_a \in (E-\delta E, E)} |c_a|^2 \langle E_a | X | E_a \rangle \quad \text{: diag.}$$

$$+ \sum_{E_a \neq E_b} c_a^* c_b e^{i(E_a - E_b)t} \langle E_a | X | E_b \rangle \quad \text{: off-diag.}$$

- Diag. part

$$\sum_{E_a \in [E-\delta E, E]} |\Delta a|^2 \langle E_a | X | E_a \rangle$$

$$\| \text{if } \theta_a, \langle E_a | X | E_a \rangle \stackrel{\text{r.l.}}{=} \langle X \rangle_{\text{eq.}}$$

$$\langle X \rangle_{\text{eq.}} := \frac{1}{w(E)} \sum_{E_a \in [E-\delta E, E]} \langle E_a | X | E_a \rangle$$

- Off-diag. part (fluctuation term)

$$\langle E_a | X | E_b \neq a \rangle \stackrel{\text{r.l.}}{=} 0$$

ETH guarantees thermalization of the system  
in any initial state  $|\Psi(0)\rangle$ .

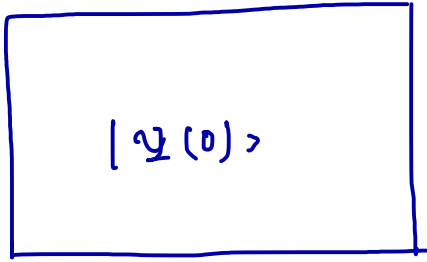
Questions)

Does strong ETH true for any quantum system?

If not, when it is not true?

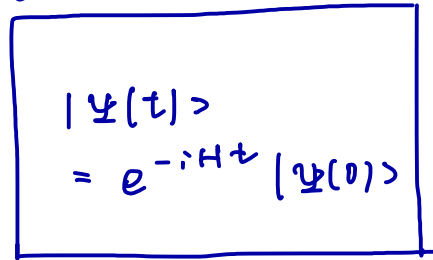
# Quantum systems with more conserved macro op.

$$\hat{Q}^{(1)} = E, \hat{Q}^{(2)}$$



→  
 $t \gg 1$

$$\hat{Q}^{(1)} = E, \hat{Q}^{(2)}$$

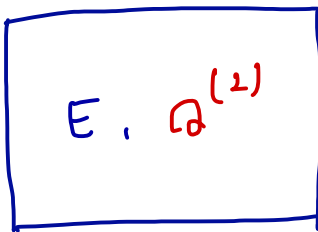


$$[\hat{Q}^{(2)}, \hat{H}] = 0$$

⊙  $\langle \psi(t) | \hat{Q}^{(2)} | \psi(t) \rangle$   
 $= \langle \psi(0) | e^{i\hat{H}t} \hat{Q}^{(2)} e^{-i\hat{H}t} | \psi(0) \rangle$   
 $= \langle \psi(0) | \hat{Q}^{(2)} | \psi(0) \rangle$

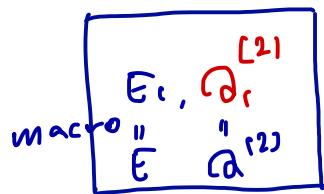
Principle of equal prob. must be stated for

equilibrium state



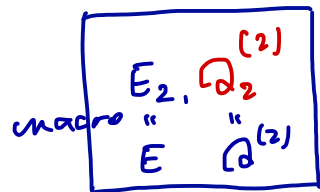
↔

energy eigenvect 1



w.p.  $\frac{1}{\Omega(E, \hat{Q}^{(2)})}$

energy eigenvect 2



w.p.  $\frac{1}{\Omega(E, \hat{Q}^{(2)})}$

⋮



Unless  $Q_2 = f(H)$ , macroscopic # of  $|E_i\rangle$  violate strong ETH:

$$\langle E_i | Q_2 | E_i \rangle \neq \langle Q_2 \rangle_{MC(E)}^{\text{macro}}$$

$$:= \frac{1}{W(E)} \sum_{E_a \in [E-\delta E, E]} \langle E_a | Q_2 | E_a \rangle.$$

Example)

$Q_2 = N$  (particle number).

$$\langle N \rangle_{MC(E)} := \frac{1}{W(E)} \sum_{E_a \in [E-\delta E, E]} \langle E_a | \hat{N} | E_a \rangle$$

$$= \frac{1}{W(E)} \sum_N \sum_{\substack{E_a \in [E-\delta E, E] \\ N_a \in [N-\delta N, N]}} \langle E_a, N_a | N | E_a, N_a \rangle$$

$$= \sum_N \frac{W(E, N)}{W(E)} \langle N \rangle_{MC(E, N)} \neq \langle N \rangle_{MC(E, N)}^{\text{macro}}$$

$$\Rightarrow \langle E_i | N | E_i \rangle \neq \langle N \rangle_{MC(E)}^{\text{macro}}$$

for macroscopic # of  $|E_i\rangle$ .

Instead,

$$\langle E_i, N_i | N | E_i, N_i \rangle = \langle N \rangle_{MC(E, N)}^{\text{macro}}$$

$$:= \frac{1}{W(E, N)} \sum_{\substack{E_a \in [E-\delta E, E] \\ N_a \in [N-\delta N, N]}} \langle E_a, N_a | N | E_a, N_a \rangle$$

Integrable systems have many conserved quantities.

$$\Omega_1 = E, \Omega_2, \Omega_3, \dots$$

$$|\Psi(0)\rangle$$

$$\xrightarrow{t \gg 1}$$

$$\Omega_1 = E, \Omega_2, \Omega_3, \dots$$

$$|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle$$

$$\forall F, [\Omega_F, H] = 0$$

We cannot define the equilibrium state since

(i)  $MC(E, \Omega_2, \Omega_3, \dots)$  ensemble can't be characterized by a few thermodynamic var.

( $E, \Omega_2, \Omega_3, \dots$  must be thermodynamic var.)

(ii)  $MC(E, \Omega_2, \Omega_3, \dots)$  average of a certain current becomes non-zero.

↳ contradicts to the def. of equilibrium state.

Questions)

- Does relaxation phenomena occur in integrable systems?
- If so, which conserved quantities are enough to describe the relaxation state?

# Integrable systems with many conserved quantities

## "Integrable Systems"

- The systems with many conserved quantities.
- No common understanding of quantum integrability.
- Sufficient condition: Yang-Baxter eq.

$$R : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V}$$

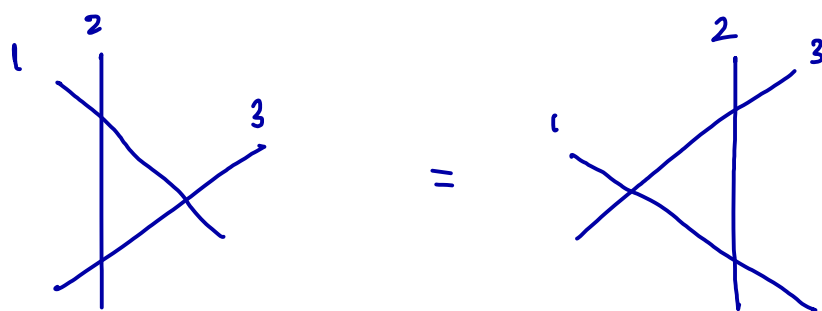
$$R_{12}, R_{13}, R_{23} : \mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}$$

$$\begin{array}{ccc} \text{"} & & \text{"} \\ R \otimes I & & I \otimes R \end{array}$$

$$R_{12}(\lambda_1 - \lambda_2) R_{13}(\lambda_1 - \lambda_3) R_{23}(\lambda_2 - \lambda_3)$$

$$= R_{23}(\lambda_2 - \lambda_3) R_{13}(\lambda_1 - \lambda_3) R_{12}(\lambda_1 - \lambda_2)$$

$$(\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C})$$



Sufficient condition that decomposition of many-body scattering does not depend on the way to decompose.

Many conserved quantities

← Commuting transfer matrices

②

← Yang-Baxter eq. (YBE)

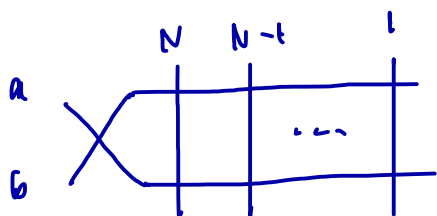
①  $M_a(\lambda) \in \text{End}(\sigma_a \otimes \sigma^{\otimes N})$  : monodromy matrix  
 $\vdots$

$$R_{a_1}(\lambda) \cdots R_{a_2}(\lambda) R_{a_1}(\lambda)$$

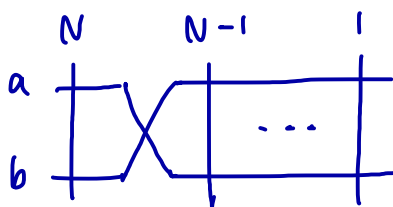
$$R_{ab}(\lambda - \mu) M_a(\lambda) M_b(\mu)$$

$$= M_b(\mu) M_a(\lambda) R_{ab}(\lambda - \mu)$$

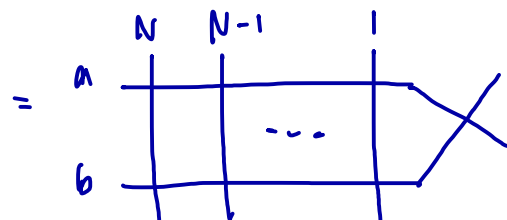
↑  
YBE



=



=



$T(\lambda) \in \text{End}(\mathcal{V}^{\otimes N})$  : transfer matrix  
 $\ddot{=}$   
 $\text{tr}_a M_a(\lambda)$

$$\text{tr}_a \text{tr}_b \left( R_{ab}(\lambda-\mu) M_a(\lambda) M_b(\mu) R_{ab}^{-1}(\lambda-\mu) \right)$$

$$= \text{tr}_a \text{tr}_b \left( M_b(\mu) M_a(\lambda) \right)$$

$R$  is invertible

$$\Rightarrow T(\lambda) T(\mu) = T(\mu) T(\lambda).$$

② Expanding  $T(\lambda), T(\mu)$  around  $\lambda=0, \mu=0$

$$T(\lambda) = \sum_r \lambda^r X_r$$

$$T(\mu) = \sum_{r'} \mu^{r'} X_{r'}$$

$$[T(\lambda), T(\mu)] = 0 \Rightarrow [X_r, X_{r'}] = 0$$

Conventionally, we choose

$$\log T(\lambda) = \sum_r \lambda^r \underline{Q}_r, \quad Q_r =: H$$

extensive conserved quantities

$$\textcircled{!} R(0) = P : \begin{matrix} v_1 \otimes v_2 \\ \uparrow \quad \uparrow \\ \mathcal{V} \quad \mathcal{V} \end{matrix} \mapsto v_2 \otimes v_1$$

# XXZ model

$$H_{XXZ} := \sum_{n=1}^N \left( \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z \right)$$

$$\sigma_n^\alpha = \downarrow \otimes \dots \otimes \downarrow \otimes \sigma^\alpha \otimes \uparrow \otimes \dots \otimes \uparrow$$

$$\downarrow, \sigma^\alpha: \mathbb{C}^2 \rightarrow \mathbb{C}^2 \quad (\alpha = x, y, z)$$

$$\downarrow = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Periodic b.c.

$$\sigma_{N+1}^\alpha := \sigma_1^\alpha$$

- Anisotropy parameter  $\Delta$

$$\begin{pmatrix} \Delta = \cos \delta \quad (\delta \in \mathbb{R}) \Rightarrow \text{gapless excitation} \\ \Delta = \cosh \eta \quad (\eta \in \mathbb{R}) \Rightarrow \text{gapped excitation} \end{pmatrix}$$

Alternatively,

$$\Delta = \frac{1}{2} (b + b^{-1}) \leftarrow "b" \text{ of } U_q(\mathfrak{sl}_2).$$

- "Ladder" operators  $e^{i\delta}$  for gapless regime  
 $e^\eta$  for gapped regime.

$$\sigma^\pm := \sigma^x \pm i \sigma^y$$

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

# R-matrix

The solution of the Yang-Baxter eq.

Ex. )

$$R(\lambda) = \begin{pmatrix} \frac{1}{2}(q e^\lambda - q^{-1} e^{-\lambda}) & & & \\ & \frac{1}{2}(e^\lambda - e^{-\lambda}) & \frac{1}{2}(q - q^{-1}) & \\ & \frac{1}{2}(q - q^{-1}) & \frac{1}{2}(e^\lambda - e^{-\lambda}) & \\ & & & \frac{1}{2}(q e^\lambda - q^{-1} e^{-\lambda}) \end{pmatrix}$$

The solution associated with  $U_q(\mathfrak{sl}_2)$  :

$$\{ S^+, S^-, K^{\pm 1} \}.$$

$$[R(\lambda), X] = 0, \quad X = S^+, S^-, K^{\pm 1}.$$

$$[S^+, S^-] = \frac{K^2 - K^{-2}}{q - q^{-1}}, \quad K S^{\pm} K^{-1} = q^{\pm 1} S^{\pm}.$$

Equivalently,

$$\Delta(S^+) = S^+ \otimes K^{-2} + I \otimes S^+$$

$$\Delta(S^-) = S^- \otimes I + K^2 \otimes S^-$$

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}$$

$$\begin{aligned} R_{au}(\lambda) &= 1_a \cdot \text{sh}\left(\lambda + \frac{i\tau}{2} \sigma_n^z\right) \text{ch}\left(\frac{i\tau}{2}\right) \\ &+ \sigma_a^z \cdot \text{ch}\left(\lambda + \frac{i\tau}{2} \sigma_n^z\right) \text{sh}\left(\frac{i\tau}{2}\right) \\ &+ \sigma_a^+ \cdot \frac{1}{2}(q - q^{-1}) S_n^- \\ &+ \sigma_a^- \cdot \frac{1}{2}(q - q^{-1}) S_n^+ \end{aligned}$$

$$R_{au}(0) = \frac{1}{2}(q + q^{-1}) \underline{P_{au}}$$

permutation.

$$\log \underbrace{T(\lambda)}_r = \sum_r \lambda^r Q_r$$

$$\text{tr}_a (R_{aN}(\lambda) \dots R_{a1}(\lambda))$$

$$Q_r = \left. \frac{d}{d\lambda} \log T(\lambda) \right|_{\lambda=0} = H_{xxz}$$

## Conserved quantities

$$\bullet \log T(\lambda) = \sum_r \lambda^r Q_r$$

$$\Rightarrow \forall r, [H_{xxz}, \underbrace{Q_r}] = 0$$

conserved quantities

$$\bullet T(\lambda) = \text{tr}_a M_a(\lambda)$$

$$R_{aN}(\lambda) \dots R_{a1}(\lambda) \in \text{End} \left( \underbrace{\sigma_a \otimes (\mathbb{C}^2)^{\otimes N}}_{\mathbb{C}^{2S+1}} \right)$$

$\mathbb{C}^{2S+1}$   $S = (\text{half-integer})$

$$\log T^{(s)}(\lambda) = \sum_r \lambda^r Q_r^{(s)}$$

$\mathbb{C}^\infty$   $s \in \mathbb{C}$   
 $\exists \subset \text{generic}$

$$\Rightarrow \forall r, s, [H_{xxz}, \underbrace{Q_r^{(s)}}] = 0$$

$\uparrow$   
 YBE

conserved quantities

# Spin- $s$ ( $2s+1$ -dim.) rep. of $U_q(\mathfrak{sl}_2)$

$$K^{\pm 1} = \sum_{r=0}^{2s} q^{\pm(s-r)} \quad (r > c \text{ or } |$$

$$S^{\pm} = \sum_{r=0}^{2s-1} \frac{b^{r+1} - b^{-r-1}}{b - b^{-1}} \quad (r > c \text{ or } |$$

$$S^{\pm} = \sum_{r=0}^{2s-1} \frac{b^{2s-r} - b^{-2s+r}}{b - b^{-1}} \quad (r+1 > c \text{ or } | \text{ for (half-) integer } s.$$

$$K^{\pm 1} = \sum_{r=0}^{2s} q^{\pm(s-r)} \quad (r > c \text{ or } |$$

$$S^{\pm} = \sum_{r=0}^{2s-1} \frac{b^{r+1} - b^{-r-1}}{b - b^{-1}} \quad (r > c \text{ or } |$$

$$S^{\pm} = \sum_{r=0}^{2s-1} \frac{b^{2s-r} - b^{-2s+r}}{b - b^{-1}} \quad (r+1 > c \text{ or } | \text{ for } s \in \mathbb{C} \\ b = \text{generic.}$$

$$\langle v | v' \rangle = \delta_{v,v'}$$

$$\{ |r\rangle \}_{r=0}^{2s} \text{ spans } \mathbb{C}^{2s+1}$$

: orthonormal basis.

# Eigenvalues & eigenvectors of $H$

determines the dynamics  
(time evolution)

Transfer matrix is the generating func. of conserved quantities (including  $H$ ).

⇒ Diagonalize  $T(k)$  instead of  $H$ .

$$\text{Let } \mathcal{V}_a = \mathbb{C}^2 = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} =: |\uparrow\rangle, \begin{pmatrix} 1 \\ 0 \end{pmatrix} =: |\downarrow\rangle \right\}.$$

$$M_a(k) = \begin{pmatrix} A(k) & B(k) \\ C(k) & D(k) \end{pmatrix} \begin{matrix} |\uparrow\rangle \\ |\downarrow\rangle \end{matrix}$$

$\mathbb{C}^{\otimes N}$  with weight  $M_z$ .

$$\left( \begin{array}{l} A, D : \partial_{\mathcal{L}_{M_z}} \rightarrow \partial_{\mathcal{L}_{M_z}} \\ B : \partial_{\mathcal{L}_{M_z}} \rightarrow \partial_{\mathcal{L}_{M_z+1}} \\ C : \partial_{\mathcal{L}_{M_z}} \rightarrow \partial_{\mathcal{L}_{M_z-1}} \end{array} \right.$$

⊙ For  $S^z := \frac{1}{2} \sum_n \sigma_n^z$ ,  $[H, S^z] = 0$

$$\Rightarrow \partial_{\mathcal{L}} = \bigoplus_{M_z} \partial_{\mathcal{L}_{M_z}}, \quad S^z \underset{\partial_{\mathcal{L}_{M_z}}}{|\psi_{M_z}\rangle} = M_z \underset{\partial_{\mathcal{L}_{M_z}}}{|\psi_{M_z}\rangle}$$

YBE

$$\Rightarrow R_{ab}(\lambda - \mu) M_a(\lambda) M_b(\mu)$$

$$= M_b(\mu) M_a(\lambda) R_{ab}(\lambda - \mu)$$

tells the algebraic (commutation) relations among  $A, B, C$ , and  $D$ .

Examples)

$$A(\lambda) B(\mu) = \frac{\text{sh}(\mu - \lambda + i\tau)}{\text{sh}(\mu - \lambda)} B(\mu) A(\lambda)$$

$$- \frac{\text{sh}(i\tau)}{\text{sh}(\mu - \lambda)} B(\lambda) A(\mu)$$

$$D(\lambda) B(\mu) = \frac{\text{sh}(\lambda - \mu + i\tau)}{\text{sh}(\lambda - \mu)} B(\mu) D(\lambda)$$

$$- \frac{\text{sh}(i\tau)}{\text{sh}(\lambda - \mu)} B(\lambda) D(\mu)$$

$$[B(\lambda), B(\mu)] = 0.$$

## Diagonalization of T

Remind  $T(\lambda) \equiv \text{tr}_n M(\lambda) = A(\lambda) + D(\lambda)$ .

Trivial eigenvector of  $T(\lambda)$

$$|\psi_{M_z = \frac{N}{2}}\rangle = |\uparrow\uparrow\cdots\uparrow\rangle = \bigotimes_{w=1}^N |\uparrow\rangle_w.$$

*∴ the highest weight state.*

One of the vector  $|\psi_{M_z = \frac{N}{2}}\rangle \in \mathcal{H}_{M_z}$  is obtained as

$$|\psi_{M_z = \frac{N}{2}}\rangle = \frac{2^{N-M_z}}{\pi} \int B(\lambda) |\psi_{M_z = \frac{N}{2}}\rangle - \quad (*)$$

Remark )

$$\langle \psi_{M_z} | \psi_{M'_z} \rangle = 0 \text{ unless } M_z = M'_z.$$

The vector  $(\star)$  is the eigenvector of  $T$  if

$$\begin{aligned}
 & T(\lambda) \prod_{n=1}^M B(\lambda_n) | \psi_{M_z = \frac{N}{2}} \rangle \\
 &= (A(\lambda) + D(\lambda)) \prod_{n=1}^M B(\lambda_n) | \psi_{M_z = \frac{N}{2}} \rangle \\
 &= \left( a(\lambda) \prod_{n=1}^M \frac{\text{sh}(\lambda_n - \lambda + i\sigma)}{\text{sh}(\lambda_n - \lambda)} + d(\lambda) \prod_{n=1}^M \frac{\text{ch}(\lambda - \lambda_n + i\sigma)}{\text{sh}(\lambda - \lambda_n)} \right) \\
 &\quad \times B(\lambda_1) \cdots B(\lambda_M) | \psi_{M_z = \frac{N}{2}} \rangle
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=1}^M \left( a(\lambda_k) \frac{\text{sh}(\lambda_k - \lambda)}{\text{sh}(i\sigma)} \prod_{n \neq k} \frac{\text{sh}(\lambda_n - \lambda_k + i\sigma)}{\text{sh}(\lambda_n - \lambda_k)} \right. \\
 &\quad \left. + d(\lambda_k) \frac{\text{sh}(\lambda - \lambda_k)}{\text{sh}(i\sigma)} \prod_{n \neq k} \frac{\text{sh}(\lambda_k - \lambda_n + i\sigma)}{\text{sh}(\lambda_k - \lambda_n)} \right) \\
 &\quad \times B(\lambda_1) \cdots B(\lambda_k) \cdots B(\lambda_M) | \psi_{M_z = \frac{N}{2}} \rangle.
 \end{aligned}$$

0. -  $(\star\star)$

$a(\lambda)$  &  $d(\lambda)$  are eigenvalues of  $A(\lambda)$  &  $D(\lambda)$  on  $| \psi_{M_z = \frac{N}{2}} \rangle$ :

$$A(\lambda) | \psi_{M_z = \frac{N}{2}} \rangle = \underbrace{a(\lambda)}_{\text{eigenvalue}} | \psi_{M_z = \frac{N}{2}} \rangle = \left( \text{sh}\left(\lambda + \frac{i\sigma}{2}\right) \right)^N | \psi_{M_z = \frac{N}{2}} \rangle$$

$$D(\lambda) | \psi_{M_z = \frac{N}{2}} \rangle = \underbrace{d(\lambda)}_{\text{eigenvalue}} | \psi_{M_z = \frac{N}{2}} \rangle = \left( \text{sh}\left(\lambda - \frac{i\sigma}{2}\right) \right)^N | \psi_{M_z = \frac{N}{2}} \rangle$$

# ★★ 'Bethe equations'

$$\left( \frac{\text{sh}(\lambda_k + \frac{i\sigma}{2})}{\text{sh}(\lambda_k - \frac{i\sigma}{2})} \right)^N = \prod_{n \neq k} \frac{\text{sh}(\lambda_k - \lambda_n + i\sigma)}{\text{sh}(\lambda_k - \lambda_n - i\sigma)}, \quad k = 1, 2, \dots, M.$$

Remarks )

- Bethe states  $\prod_{a=c}^M B(\lambda_a) | \psi_{M_2 = \frac{N}{2}} \rangle$  are the h.w.s.  
 $\uparrow$   
 sol. of the Bethe eq.

$$S^+ \prod_{a=c}^M B(\lambda_a) | \psi_{M_2 = \frac{N}{2}} \rangle = 0.$$

- The other eigenvector of  $T$  (not Bethe states) are constructed by applying  $S^-$

$$\begin{aligned} S^- \left( \prod_{a=c}^M B(\lambda_a) | \psi_{M_2 = \frac{N}{2}} \rangle \right. \\ S^- \left( S^- \prod_{a=c}^M B(\lambda_a) | \psi_{M_2 = \frac{N}{2}} \rangle \right. \\ (S^-)^2 \prod_{a=c}^M B(\lambda_a) | \psi_{M_2 = \frac{N}{2}} \rangle \\ \vdots \\ S^- \left( (S^-)^{N-2M} \prod_{a=c}^M B(\lambda_a) | \psi_{M_2 = \frac{N}{2}} \rangle \right. \end{aligned}$$

(lowest weight state.)

# Bethe roots

Bethe equations

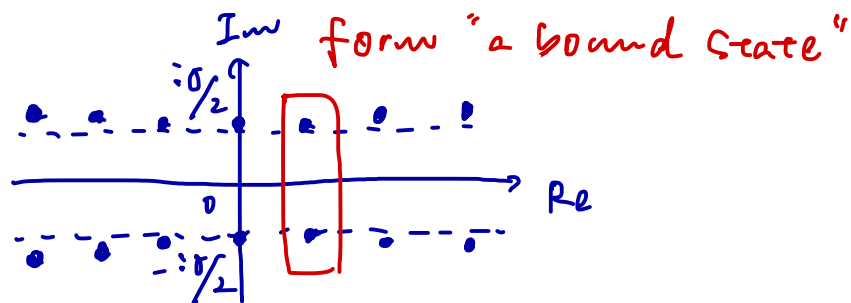
$$\left( \frac{\text{sh}(\lambda_j - \frac{i\pi}{2})}{\text{sh}(\lambda_j + \frac{i\pi}{2})} \right)^N = \prod_{k \neq j} \frac{\text{sh}(\lambda_j - \lambda_k - i\pi)}{\text{sh}(\lambda_j - \lambda_k + i\pi)}$$

are known to form "string solutions" in large  $N$ :

$$\lambda_n^{j,r} \approx \lambda_a^j + \frac{i\pi}{2} (l - 2r + 1) + \underbrace{O(e^{-N})}_{\substack{\text{string center} \\ \text{O}(e^{-N}) \text{ l-string solution.}}} \quad r = 1, 2, \dots, l$$

Examples)

2-string solutions



- $q = e^{i\theta}$ ,  $r = \frac{\pi}{l}$  ( $l$ : coprime)

( $1, 2, \dots, (l-1)$ -string sol.  
anti-string sol.)

- $b = e^\eta$

$1, 2, \dots$  string sol.

## Bethe solutions in the $N \rightarrow \infty$ limit.

Bethe eq.

$$\Rightarrow \Theta_{\text{fin}}(\lambda_j) + \frac{1}{N} \sum_{k=1}^M \Theta_{\text{scat}}(\lambda_j - \lambda_k) = \frac{2\pi}{L} \underbrace{I_j}$$

$$\left( \begin{array}{l} \Theta_{\text{fin}}(\lambda) := \frac{i}{N} \sum_{n=1}^N \ln \frac{\text{sh}(\lambda - \frac{i\pi}{2})}{\text{sh}(\lambda + \frac{i\pi}{2})} \\ \Theta_{\text{scat}}(\lambda) := i \ln \frac{\text{sh}(i\pi + \lambda)}{\text{sh}(i\pi - \lambda)} \end{array} \right. \begin{array}{l} \text{int. for odd } N \\ \text{half-int for even } N \end{array}$$

when  $\lambda_j, \lambda_j \in \mathbb{R}$ .

$\Rightarrow$  A set of (half-)integers  $\{I_j\}$  uniquely correspond to an eigenvector.

String center of each string solutions becomes dense in the thermodynamic limit.

$$N \rightarrow \infty, \quad M/N \text{ fixed.}$$

$\Rightarrow$  "Density of Bethe roots"

$$\rho(\lambda) := \frac{d}{d\lambda} \underbrace{x(\lambda)}_{\substack{\uparrow \\ x_j := \frac{I_j}{N}}}$$

for the ground state  $(\lambda_j, I_{j+1} - I_j = 1)$ .

Remark )  $W(\lambda|\rho|\tau)$

Mang microstates could be expressed by  
the same Bethe root density.

( We only discuss the macro quantities that are  
insensitive to the micro feature. )

$S_{YY} := W(\lambda|\rho|\tau) = \text{Yang - Yang entropy.}$

## § 4. Specific behavior of integrable systems

Due to existence of many conserved quantities, integrable systems show specific behaviors uncommon to normal thermalizing systems.

### Relaxation state of the XXZ model

Proposition) [Frigoli et al. 2008, CM 2020]

$$\rho_{\text{GGE}} = \frac{1}{Z} e^{-\sum_r \text{tr} \rho_r}$$

The XXZ model relaxes to the density matrix

$$\rho_{\text{GGE}} = \frac{1}{Z} e^{-\sum_{n=1}^{\ell-1} \left( \sum_{r=1}^r h_n^r \rho_n^r + h_n^s \rho_{n,p}(s) \right)}, \quad p \in \mathbb{Z}_{\geq 0}$$

$(q = e^{i\frac{\pi}{2} \times \frac{u}{\ell}})$

in the thermodynamic limit.



The relaxation state of the XXZ model in TL is completely described by  $\underbrace{\{ \rho^{(r)}(\lambda_{\text{th}}) \}_{r=1, \dots, \ell}}_4$ .

$$\{ \rho_i^{(r)}(\lambda_{\text{th}}) \}_{r=1, \dots, \ell-1} \wedge \{ \rho_{s,0}(\lambda_{\text{th}}, s) \}$$

Linearly-independent set of conserved quantities.

# $X \times Z$ model

• eigenvectors for  $V \rightarrow a$

$$\approx \{ \rho^r(\lambda) \}_r$$

+

saddle point method

$\{ \rho^r(\lambda) \}_r$  for  
the steady state.

$$\cdot \{ \rho^r(\lambda) \}_v \approx \{ T^r(\lambda) \}_v \approx \{ Q^r \}_r$$

transfer  
matrices

conserved  
quantities

Conserved quantities in terms of  $\rho^{(r)}$

$$\log \underbrace{T^{(r)}(\lambda)}_{\text{ii}} = \sum_n (\lambda - \lambda_0)^n \rho_n^{(r)}(\lambda_0)$$

$$\text{tr}_a M_a^{(r)}(\lambda), M_a^{(r)}(\lambda) \in \text{End} \left( \underbrace{\mathbb{C}^2}_{\mathbb{C}} \otimes \underbrace{(\mathbb{C}^2)^{\otimes N}}_{\mathbb{C}^{2r+1}} \right)$$

$$\rho_1^{(r)}(\lambda_0) \underset{\text{T.L.}}{\propto} \sum_{i=1}^L \sum_{a=1}^{\min(n_i, r)} \theta_{|r-n_i|-1+2a, n_i}^{(1,0)} * \underbrace{\rho_j^s(\lambda_0)}_{\text{convolution}}$$

$$\log \underbrace{T(\lambda, s)}_{\text{ii}} = \sum_{n, m} (\lambda - \lambda_0)^n (s - s_0)^m \rho_{n, m}(\lambda_0, s_0)$$

$$\text{tr}_a M_a(\lambda, s), M_a(\lambda, s) \in \text{End} \left( \underbrace{\mathbb{C}^2}_{\mathbb{C}} \otimes \underbrace{(\mathbb{C}^2)^{\otimes N}}_{\mathbb{C}^\infty \text{ (spin-} \frac{1}{2} \text{ rep.)}} \right)$$

$\rho_{n, m}(\lambda, s)$

$$\underset{\text{T.L.}}{\propto} \sum_{i=1}^L \sum_{a=1}^{n_i} \theta_{2s - \frac{k-1}{2} - n_i - 1 + 2a, n_i}^{(n, m)} * \underbrace{\rho_j^s \left( \lambda - i \left( \frac{k-1}{4} + \frac{1}{2} \right) \delta \right)}_{\text{convolution}}$$

$$- \sum_{i=1}^L \sum_{a=1}^{\min(n_i, \frac{k-1}{2})} \theta_{|\frac{k-1}{2} - n_i| - 1 + 2a, n_i}^{(n, m)} * \underbrace{\rho_j^s \left( \lambda + i \left( \frac{k-1}{4} - s - \frac{1}{2} \right) \delta \right)}_{\text{convolution}}$$

# The fermions

$$\theta_{n,v}^{(c,0)}(\lambda) = \frac{v}{\pi} \frac{\sin(n\lambda)}{\text{ch}(2\lambda) - v \cos(n\lambda)}$$

$$\theta_{n,v}^{(0,1)}(\lambda) = \frac{v\lambda}{\pi} \frac{\text{sh}(2\lambda)}{\text{ch}(2\lambda) - v \cos(n\lambda)}$$

$$v = \begin{cases} +1 & \text{for string} \\ -1 & \text{for anti-string} \end{cases}$$

Satisfy

$$\sum_{a=1}^{\min(n_j, 2S+1)} \text{FF} \left[ \theta_{|2S+1-n_j|-1+2a, v_j}^{(c,0)}(\lambda) \right]$$

$$+ \sum_{a=1}^{\min(n_j, 2S-1)} \text{FF} \left[ \theta_{|2S-1-n_j|-1+2a, v_j}^{(c,0)}(\lambda) \right]$$

$$= 2 \cosh\left(\frac{\pi k}{2l}\right) \sum_{a=1}^{\min(n_j, 2S)} \text{FF} \left[ \theta_{|2S-n_j|-1+2a, v_j}^{(c,0)}(\lambda) \right] - \delta_{j, 2S}$$

Fourier transf.

$$\text{FF}[\rho(\lambda)] := \int_{-\infty}^{\infty} d\lambda e^{-iF\lambda} \rho(\lambda)$$

Proposition )

Conserved quantities are expressed by the densities of string centers.

"String-charge duality"

$$\begin{aligned} \bullet F_k[\rho^{(r)}(\lambda)] - \delta_{n, l-1} F_k[\rho^{(l)}(\lambda)] \\ = 2 \operatorname{ch}\left(\frac{\sigma F}{2}\right) F_k[\mathcal{Q}_i^{(r)}(\lambda)] - F_k[\mathcal{Q}^{(n+1)}(\lambda)] \\ - F_k[\mathcal{Q}^{(r-1)}(\lambda)] \\ (n = 1, 2, \dots, l-1) \end{aligned}$$

$$\begin{aligned} \bullet F_k[\rho^{(l)}(\lambda)] = - \frac{\operatorname{sh}\left((l-2s)\frac{\sigma F}{2}\right)}{\operatorname{ch}\left((l-2s-1)\frac{\sigma F}{2}\right)} F_k[\mathcal{Q}_1^{(l-1)}(\lambda)] \\ + \frac{\operatorname{sh}\left(\frac{\sigma F}{2}\right)}{\operatorname{sh}\left((l-2s-1)\frac{\sigma F}{2}\right)} F_k[\mathcal{Q}_{(1,0)}(\lambda, s)] \end{aligned}$$

Lemma )

Spin-flip non-invariant changes are not linearly independent.

$$F_k [ \mathcal{Q}_{(1,0)}(\lambda, s) ] = \frac{\text{sh} \left( (l-2s-1) \frac{\sigma k}{2} \right)}{\text{sh} \left( (l-2t-1) \frac{\sigma k}{2} \right)} F_k [ \mathcal{Q}_{(1,0)}(\lambda, t) ] \\ - \frac{\text{sh} \left( (2t-2s) \frac{\sigma k}{2} \right)}{\text{sh} \left( (l-2t-1) \frac{\sigma k}{2} \right)} F_k [ \mathcal{Q}_1^{(l-1)}(\lambda) ]$$

$$F_k [ \mathcal{Q}_{r,2p}(\lambda, s) ] = (-ik)^{r-1} (-\sigma k)^{2p} F_k [ \mathcal{Q}_{(1,0)}(\lambda, s) ]$$

$$F_k [ \mathcal{Q}_{r,2p-1}(\lambda, s) ] = (-ik)^{r-1} \frac{(-\sigma k)^{2p-1}}{\text{sh} \left( (l-2s-1) \frac{\sigma k}{2} \right)} F_k [ \mathcal{Q}_1^{(l-1)}(\lambda) ]$$

$$= (-ik)^{r-1} (-\sigma k)^{2p-1} \coth \left( (l-2s-1) \frac{\sigma k}{2} \right) F_k [ \mathcal{Q}_{(1,0)}(\lambda, s) ]$$

# Non-zero (macroscopic) spin current

Drude weight (in linear response)

$$D(\beta = \frac{1}{k_B T}) = \lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{\beta}{2Nt} \int_0^t dt' \langle J(0), J(t') \rangle_\beta$$
$$= \lim_{N \rightarrow \infty} \frac{1}{2N} \frac{|\langle J, \mathcal{O}_{R, 2p} \rangle|^2}{\|\mathcal{O}_{R, p}\|_\beta^2}$$

$D > 0 \Rightarrow$  ballistic transport  
(non-vanishing DC current)

Prop.) Drude weight for spin current is non-zero at high temperature.

$$\lim_{\beta \rightarrow 0} D_S(\beta) > 0$$

(\*)

Conserved operator constructed from the monodromy matrix with the aux. space of complex-spin rep. has finite overlap with the spin current operator

$$\frac{i}{2} \sum_n (\sigma_n^x \sigma_{n+1}^x - \sigma_n^y \sigma_{n+1}^y)$$

$$\left( \begin{array}{l} \langle J_S, \mathcal{O}_{R, 2p-1}(\lambda, s_0) \rangle \sim N^{r+p} \\ \langle J_S, \mathcal{O}_{R, 2p}(\lambda, s_0) \rangle \sim 0 \\ \|\mathcal{O}_{R, p}(\lambda, s_0)\|^2 \sim N^{2r+p} \end{array} \right.$$

## § 4 Summary

- Integrable systems have many conserved quantities due to the Yang-Baxter eq.
- Equilibrium state can't be defined for integrable systems (equivalently, integrable systems do not thermalize).

Instead, integrable systems relax to the steady state called "generalized Gibbs ensemble (GGE)".

- The XXZ model (an example of integrable systems) shows finite spin current even after relaxation.