Hecke algebras, KLR algebras, and James's conjecture

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Hecke algebras

The type A Hecke algebra is the unital, associative $\mathbb{F}$-algebra $\mathscr{H}_n$ with generators $T_1, T_2, \dots, T_{n-1}$ and relations

$$\begin{aligned} (T_i - q)(T_i + 1) &= 0 && \text{for all } i, \\ T_i T_j &= T_j T_i && \text{for } |i-j| > 1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \leqslant i \leqslant n-2, \end{aligned}$$

where $q \in \mathbb{F}$ is a primitive eth root of unity or $q = 1$.
N.B. setting $q = 1 \rightsquigarrow \mathscr{H}_n \cong \mathbb{F}\mathfrak{S}_n$. $\mathscr{H}_n$ is semisimple if $e > n$.

The *Specht modules* $\{\mathrm{S}^\lambda \mid \lambda \vdash n\}$ over $\mathscr{H}_n$ are the ordinary irreducible $\mathscr{H}_n$-modules, indexed by partitions λ of n.

If $e \leqslant n$, the simple modules appear as quotients of the Specht modules: $\{\mathrm{D}^\lambda \mid \lambda \vdash n,\ \lambda \text{ is } e\text{-restricted}\}$.

(λ is e-restricted means $\lambda_i - \lambda_{i+1} < e$ for all i.)

Main problem

Open Problem

Determine the dimensions of simple modules D^λ.

N.B. the dimensions of Specht modules are well-known.

Decomposition numbers

A related problem is to determine the composition multiplicities of simple modules D^μ in Specht modules S^λ. We define $d^p_{\lambda\mu}$ to be the composition multiplicity of D^μ in S^λ. In other words,

$$d^p_{\lambda\mu} = [\mathrm{S}^\lambda : \mathrm{D}^\mu] \in \mathbb{N}.$$

Since we know the dimensions of Specht modules, and some unitriangularity results for $d^p_{\lambda\mu}$, knowing the decomposition numbers would tell us the dimensions of simple modules.

Main problem

When $p = 0$, Ariki's categorification theorem tells us that the decomposition numbers are equal to the canonical basis coefficients in $V(\Lambda_0)$, a highest weight irreducible module over $U_q(\widehat{\mathfrak{sl}_e})$, i.e. $d^0_{\lambda\mu}$ is the coefficient of λ in the canonical basis vector $G(\mu)$.

Lascoux, Leclerc, and Thibon produced an algorithm, recursive in $n = |\lambda|$, that computes these coefficients.

Upshot

In characteristic 0, we can recursively compute all of the decomposition numbers for $\mathscr{H}_n$ by the LLT algorithm.

Adjustment matrices

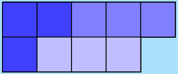
Let D^p denote the decomposition matrix $(d^p_{\lambda\mu})_{\lambda,\mu}$. There exists a square, unitriangular matrix A^p such that $D^p = D^0 A^p$.

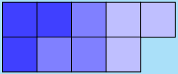
Blocks

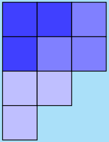
Specht modules S^λ and S^μ (or simple modules D^λ and D^μ) are in the same block of $\mathscr{H}_n$ if and only if λ and μ *have the same e-core*.

Example

Let $\lambda = (5, 4)$, $\mu = (3^2, 2, 1)$, and $e = 3$. Then λ and μ are in the same block:

$[\lambda] =$ 



$[\mu] =$ 

The *weight* or *defect* of a partition is the number of e-rim hooks that can be removed before obtaining the core. e.g. $w = 3$ above, with core the empty partition.

Blocks

An equivalent description can be given in terms of partitions having *equal multisets of residues modulo e*. These are column number − row number taken modulo e.

Example

Let $\lambda = (5, 4)$, $\mu = (3^2, 2, 1)$, and $e = 3$. Then λ and μ are in the same block:

$[\lambda] =$

0	1	2	0	1
2	0	1	2	

$[\mu] =$

0	1	2
2	0	1
1	2	
0		

James's conjecture

Recall the factorisation $D^p = D^0 A^p$. The LLT algorithm allows us to compute D^0, so that in a sense, A^p encodes the difficult part of the problem of computing decomposition numbers.

James's conjecture, 1990

If $n < ep$, then A^p is the identity matrix.

James's conjecture, blockwise refinement

Fix a block of $\mathscr{H}_n$ of weight w. If $w < p$, then A^p is the identity matrix.

The stronger, blockwise version is known to be true for blocks of weight $w \leqslant 4$, by works of Richards and of Fayers. It is also known to be true for the principal blocks of $\mathscr{H}_{5e}$ by work of Low, and for all 'RoCK block', by work of James, Lyle and Mathas.

Williamson's counterexample

Williamson's counterexample to James's conjecture

In 2013, Geordie Williamson showed that there are counterexamples to James's conjecture. The smallest counterexample produced by his methods occurs in the principal block of $\mathbb{F}\mathfrak{S}_n$ (i.e. $e = p$) for $n = 1\,744\,860$, in characteristic $p = 2237$.

The order of this $\mathfrak{S}_n$ is a number with $10,133,219$ digits. Moreover, Williamson does not provide the adjustment matrix or decomposition numbers in this example, or even say which simple module's dimension changes in characteristic p.

KLR algebras – motivation

Let $\mathfrak{g}$ be a symmetrisable Kac–Moody algebra (e.g. a finite-dimensional or affine type Lie algebra).

The Khovanov–Lauda–Rouquier (KLR) algebras $\mathscr{R}_\beta(\mathfrak{g})$ and their cyclotomic quotients $\mathscr{R}_\beta^\Lambda(\mathfrak{g})$ **categorify** the negative halves of the quantum groups $U_q(\mathfrak{g})$ and their irreducible highest weight modules $V(\Lambda)$, respectively. These algebras are $\mathbb{Z}$-graded.

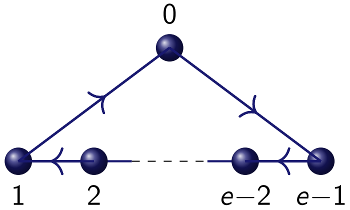
Theorem (Brundan–Kleshchev, 2009)

If $\mathfrak{g}$ is of type A_∞ or $A_{e-1}^{(1)}$, then each block of an 'integral cyclotomic Hecke algebra' is isomorphic to a cyclotomic KLR algebra $\mathscr{R}_\beta^\Lambda(\mathfrak{g})$. **Special case: blocks of $\mathscr{H}_n$ are isomorphic to some $\mathscr{R}_\beta^{\Lambda_0}(\mathfrak{g})$.**

Upshots – type A KLR algebras are better understood than other types; $\mathscr{H}_n$ has a $\mathbb{Z}$-grading, and we may study its **graded** representation theory.

KLR algebras – definitions

$\mathscr{H}_n$ is semisimple for $e = \infty$; equivalently $\mathscr{R}_\beta^{\Lambda_0}(\mathfrak{g})$ is semisimple in type A_∞. So we'll focus on type $A_{e-1}^{(1)}$, which has the following quiver.



KLR algebras – definitions

Let $I = \{0, 1, \ldots, e-1\}$ be the vertex set of the underlying quiver ($\leftrightsquigarrow$ simple roots $\{\alpha_i \mid i \in I\}$). Let $Q^+ := \bigoplus_{i \in I} \mathbb{Z}_{\geqslant 0}\alpha_i$ be the positive cone of the root lattice. Let $\beta \in Q^+$, and define

$$I^\beta = \{\mathbf{i} \in I^n \mid \alpha_{i_1} + \cdots + \alpha_{i_n} = \beta\}.$$

Then the KLR algebra $\mathscr{R}_\beta$ is the unital associative $\mathbb{F}$-algebra with generators

$$\{e(\mathbf{i}) \mid \mathbf{i} \in I^\beta\} \cup \{y_1, \ldots, y_n\} \cup \{\psi_1, \ldots, \psi_{n-1}\},$$

subject to many relations.

KLR algebras – definitions

This $\mathbb{Z}$-graded algebra is infinite-dimensional. For $\Lambda = \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_r}$ a dominant integral weight, the **cyclotomic** KLR $\mathscr{R}^{\Lambda}_{\beta}$ is the quotient of $\mathscr{R}_{\beta}$ by the extra relations

$$y_1^{\langle \alpha_{i_1}^{\vee}, \Lambda \rangle} e(\mathbf{i}) = 0 \text{ for all } \mathbf{i} \in I^{\beta}.$$

Special case: $\Lambda = \Lambda_0$, the relation becomes

$$\boldsymbol{y_1 = 0 \text{ and } e(\mathbf{i}) = 0 \text{ if } i_1 \neq 0.}$$

Specht modules – Tableaux

Given a partition λ, a λ-tableau is a filling of the Young diagram with the numbers $1, \ldots, n$ without repeats. A tableau $\mathtt{T}$ is **standard** if its entries increase along rows and down columns. The most dominant standard λ-tableau, denoted $\mathtt{T}^\lambda$, is formed by putting entries in order along the first row of the first component, then the second, etc.

Example

Let $\lambda = (4, 3, 1)$, a partition of 8. Then

$$\mathtt{T}^\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \cline{1-3} 8 & & & \\ \cline{1-1}\end{array}, \quad \mathtt{S} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 7 \\ \hline 2 & 4 & 8 & \\ \cline{1-3} 5 & & & \\ \cline{1-1}\end{array}$$

are two standard λ-tableaux.

Specht modules – Residues

Any standard λ-tableau $\mathtt{T}$ can be written as $\mathtt{T} = w^{\mathtt{T}}\mathtt{T}^{\lambda}$ for some permutation $w^{\mathtt{T}}$. For each standard λ-tableau $\mathtt{T}$, fix a reduced expression $w^{\mathtt{T}} = s_{i_1} \dots s_{i_k} \rightsquigarrow \psi^{\mathtt{T}} = \psi_{i_1} \dots \psi_{i_k}$.

We also assign a residue sequences to $\mathtt{T}$. The node (r, c) of a Young diagram $[\lambda]$ carries residue $c - r \pmod e$, and we read off residues in order of the entries of $\mathtt{T}$ to give its residue sequence $i^{\mathtt{T}}$.

Example

Let $\lambda = (4, 3, 1)$, and $e = 3$. Then

$[\lambda] =$

0	1	2	0
2	0	1	
1			

, $\mathtt{S} =$

1	3	6	7
2	4	8	
5			

so that $i^{\mathtt{S}} = (0, 2, 1, 0, 1, 2, 0, 1)$.

Specht modules – Presentation and basis

Let λ have content $\beta \in Q^+$. The Specht module S^λ is the graded $\mathscr{R}_\beta^{\Lambda_0}$-module generated by the vector v^λ subject to the relations:

- $e(\mathbf{i})v^\lambda = \delta_{\mathbf{i},\mathbf{i}^{\mathtt{T}^\lambda}} v^\lambda$ for all $\mathbf{i} \in I^\beta$; $\rightsquigarrow e(\mathbf{i})v^{\mathtt{T}} = \delta_{\mathbf{i},\mathbf{i}^{\mathtt{T}}} v^{\mathtt{T}}$.
- $y_r v^\lambda = 0$ for all r;
- $\psi_r v^\lambda = 0$ whenever r and $r+1$ are adjacent in $\mathtt{T}^\lambda$;
- $g^u v^\lambda = 0$ for each $u \in [\lambda]$ with a node below it in $[\lambda]$ (i.e. for each 'Garnir node' u).

Theorem (Kleshchev–Mathas–Ram, 2012)

S^λ *has a homogeneous basis* $\{v^{\mathtt{T}} = \psi^{\mathtt{T}} v^\lambda \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$.

The Gram matrix

The basis above also arises via a 'graded cellular basis' of $\mathscr{R}_\beta^{\Lambda_0}$. General theory then gives that the Specht module S^λ is equipped with a $\mathbb{Z}$-valued homogeneous degree zero bilinear form $\langle -, - \rangle$ arising from the basis above, and determined by

$$\langle v^{\mathtt{S}}, v^{\mathtt{T}} \rangle v^\lambda = e(\mathbf{i}^\lambda) y^\lambda (\psi^{\mathtt{S}})^* v^{\mathtt{T}}.$$

We define $\operatorname{rad} \mathsf{S}^\lambda = \{x \in \mathsf{S}^\lambda \mid \langle x, y \rangle = 0 \text{ for all } y \in \mathsf{S}^\lambda\}$, and then $\mathsf{D}^\lambda = \mathsf{S}^\lambda / \operatorname{rad} \mathsf{S}^\lambda$ – this is a simple module if λ is e-restricted, and 0 otherwise.

The **Gram matrix** is $(\langle v^{\mathtt{S}}, v^{\mathtt{T}} \rangle)_{\mathtt{S},\mathtt{T} \in \mathrm{Std}(\lambda)}$, and its rank gives the dimension of D^λ.

Graded decomposition numbers

The **graded** decomposition number $d^p_{\lambda\mu}(v)$ is defined to be the **graded** composition multiplicity of D^μ in S^λ. In other words,

$$d^p_{\lambda\mu}(v) = [\mathsf{S}^\lambda : \mathsf{D}^\mu]_v = \sum_{d \in \mathbb{Z}} [\mathsf{S}^\lambda : \mathsf{D}^\mu \langle d \rangle] v^d \in \mathbb{N}[v, v^{-1}].$$

The Gram matrices from the previous slide may be graded, and also cut into individual 'weight spaces' $e(\mathbf{i})\,\mathsf{S}^\lambda$ so that computing the ranks actually gives us the graded dimension of each weight space $e(\mathbf{i})\,\mathsf{D}^\lambda$.

Where to find new counterexamples?

Since James's conjecture holds for blocks of weight < 5, and for the principal blocks of $\mathscr{H}_{5e}$, any counterexamples would have to lie in some $\mathscr{H}_n$ for $n > 5e$ (but hopefully significantly smaller than 1 744 860).

We can even work blockwise, and ignore all blocks of weight < 5. Even so, it is a needle in a haystack problem, and $n > 5e$ already forces you to look at fairly large algebras!

But, since the ranks of Gram matrices tell us the dimensions of simple modules, they are a tool we may use to attempt to find such counterexamples – we're looking for simple modules whose dimensions change in characteristic $p > w$ vs. in characteristic 0.

An explicit counterexample

Letting $e = 4$ and $n = 24$, we may take the principal block of $\mathscr{H}_n = \mathscr{H}_{6e}$. Let $\lambda = (5, 4, 3^2, 2, 1^7)$, and look at the Specht module S^λ. In particular, look at the weight space

$$e(0,3,2,1,0,3,2,1,0,3,2,1,0,3,2,1,0,3,2,1,0,3,2,1)\,\mathrm{S}^\lambda,$$

which is the only weight space in the sign module $\mathrm{S}^{(1^n)} = \mathrm{D}^{(1^n)}$. This weight space has graded dimension $9v^4 + 43v^2 + 19$. Since there are no terms of degree -4 or -2, we know that those positive degree parts all lie in the radical, in any char. So we examine the 19-dimensional part of this weight space that is in degree 0. We compute its Gram matrix. (It takes about two hours to compute this on my laptop!)

The Gram matrix

We discover that the Gram matrix is

$$\begin{bmatrix}
-2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -2 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -2 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 1 & -2 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -2 & 2 & 0 & -4 & 0 & 0 & 1 & -1 & 0 & 2 & 0 & 0 & -1 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & -1 & 2 & 0 & -4 & 0 & 0 & 2 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & -4 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & -4
\end{bmatrix}$$

What now?

The conclusion

In GAP, one can quickly check that this matrix has elementary divisors 1 (with multiplicity 12) and 14 (with multiplicity 1). In other words, the matrix has rank 12 when working in characteristic 2 or 7, but has rank 13 otherwise (in particular, in char. $p = 0$).

So in char. 0, the degree 0 part of this weight space of D^λ is 13-dimensional, while in char. 7, it is only 12-dimensional. This comes from S^λ having an extra simple module in the radical in char. 7, i.e. a decomposition number is increasing, so the adjustment matrix non-trivial.

But the weight of the block is only 6, so we have $p > w$, and this is a counterexample to James's conjecture **in rank 24**. (N.B. the dimension of $\mathscr{H}_n$ is 24!, a 24 digit number – large, but still an incredible improvement!)

Final comments

We can do slightly better, and show that

$$d^0_{\lambda(1^n)} = 3v^2 \neq 3v^2 + 1 = d^7_{\lambda(1^n)}.$$

We can also find bigger counterexamples, including one in $\mathbb{F}\mathfrak{S}_{44}$. But the above example is the smallest rank one for $e > 2$. For $e = 2$, Mathas has verified that no counterexample exists for $n < 18$. But in all counterexamples we can generate, the weight is at least 6.

Conjecture

James's conjecture holds for blocks of weight 5 or less.
(Fix a block of $\mathscr{H}_n$ of weight w. If $w < p$ and $w \leqslant 5$, then A^p is the identity matrix.)

We also have no counterexamples if $e = 2$ or 3, so it's tempting to expect James's conjecture to hold for **all** blocks in those cases.