GENERAL RELATIVITY HOMEWORK – WEEK 6

As we saw, in 3+1 dimensions, GR recovers Newtonian gravity. In 2+1 dimensions, things are very different, as we'll now explore. In this exercise set, all indices are internal. Capital indices I take values $I = t, x, y$, while lowercase indices take the spatial values $i = x, y$. The flat internal metric is $\eta_{IJ} = \mathrm{diag}(-1, 1, 1)$ (with the obvious addition of one more spatial axis when discussing 3+1d).

Exercise 1. *We mentioned in the lecture that in 2+1d, the Riemann tensor R_{IJKL} is fully described by the Ricci tensor R_{IJ}. This means that in 2+1d, there are no locally observable effects of gravity in vacuum! Let's demonstrate this statement:*

1. *What are the independent components of R_{IJKL} in 2+1d?*
2. *Express the components of R_{IJ} in terms of the components of R_{IJKL}.*
3. *Invert this relationship to express R_{IJKL} in terms of R_{IJ}.*

Exercise 2. *In spacetime of any dimension, the Einstein equations read:*

$$R_{IJ} - \frac{1}{2} R \, \eta_{IJ} = \kappa T_{IJ} \ , \tag{1}$$

where κ is a conveniently rescaled version of the gravitational constant (in 3+1d, $\kappa = 8\pi G$).

1. *From the universal form* (1) *of the Einstein equations, derive the version we quoted in class for 3+1d:*
$$R_{IJ} = \kappa \left(T_{IJ} - \frac{1}{2} T^K_K \eta_{IJ} \right) \ . \tag{2}$$
2. *What is the analogue of* (2) *in 2+1d?*
3. *Consider a non-relativistic stress-energy tensor $T_{IJ} = \mathrm{diag}(\rho, 0, 0)$ in 2+1d. Find the resulting Ricci tensor R_{IJ}.*
4. *Using the result of Exercise 1, find the Riemann tensor R_{IJKL} in the above non-relativistic case. What do you get for the acceleration gradient R_{tijt}?*