GENERAL RELATIVITY HOMEWORK – WEEK 5

Exercise 1. *Consider the sphere – same as in Q1 of the midterm, but with a general radius r. The coordinates are $x^\mu = (\theta, \phi)$. The internal-space indices take values $I = \hat{\theta}, \hat{\phi}$, with Euclidean internal metric δ_{IJ}. The non-vanishing components of the vielbein are:*

$$e_\theta^{\hat{\theta}} = r \; ; \quad e_\phi^{\hat{\phi}} = r \sin\theta \; . \tag{1}$$

1. *Find the components of the spin-connection ω_μ^{IJ}.*
2. *Find the (single, because we're in 2d) independent component of the curvature $F_{\mu\nu}^{IJ}$ and Riemann tensor R_{IJKL}.*
3. *A vector on the sphere is parallel-transported along an infinitesimal loop of area A, and ends up rotated by angle α. From the value of the Riemann tensor, deduce the proportionality relation between A and α (no calculation should be needed).*

Exercise 2. *Consider three (non-infinitesimal) closed loops along the sphere from Q1:*

1. *From $(\theta = \pi/2, \phi = 0)$ to $(\theta = 0)$ along the meridian, then to $(\theta = \pi/2, \phi = \pi/2)$ along a different meridian, then back to $(\theta = \pi/2, \phi = 0)$ along the equator.*
2. *From $(\theta = \pi/2, \phi = 0)$ to $(\theta = \pi/2, \phi = \pi)$ along the meridian, then back to $(\theta = \pi/2, \phi = 0)$ along the equator.*
3. *From $(\theta = \pi/2, \phi = 0)$ back to itself along the equator.*

Find the rotation angle α of a vector parallel-transported along each of these loops (no calculation should be needed, you can do this with your fingers). What is the area A of the sphere's surface enclosed in each loop? Compare with the proportionality relation from Q1.

Exercise 3. *Now, let's upgrade the constant radius r to a third coordinate, and add a third internal-space axis $\hat{r}$. The Euclidean internal metric is now the 3×3 identity matrix δ_{IJ}, and the non-vanishing components of the vielbein are:*

$$e_r^{\hat{r}} = 1 \; ; \quad e_\theta^{\hat{\theta}} = r \; ; \quad e_\phi^{\hat{\phi}} = r \sin\theta \; . \tag{2}$$

1. *Find the components of the spin-connection ω_μ^{IJ}.*
2. *Find the components of the curvature $F_{\mu\nu}^{IJ}$. Interpret the result geometrically – what space is this?*