GENERAL RELATIVITY FINAL EXAM: THE LIGHT WEIGHS HEAVY

Part I: Falling light

Question 1. *In Newtonian gravity, consider a small non-relativistic particle flying at velocity v past a point mass M, situated at $\mathbf{r} = 0$. Assume that the particle is deflected only slightly by the gravitational field of M. Thus, its trajectory is nearly a straight line along e.g. the z axis. Let b denote the "impact parameter", i.e. the trajectory's displacement from the origin along e.g. the x axis. Find the small angle ϕ by which the gravity of M deflects the particle, by integrating the x component of the acceleration.*

Question 2. *In Special Relativity, write down the stress-energy tensor $T^{\mu\nu}$ describing a point mass M. Now, consider the resulting linearized perturbation $h_{\mu\nu} = (e^I_\mu - \delta^I_\mu)\eta_{I\nu}$ of the geometry. By rotational symmetry and the freedom to set t orthogonal to r, its non-vanishing components must take the form:*

$$h_{tt} = A(r) \ ; \quad h_{ij} = B(r)\hat{r}_i\hat{r}_j + C(r)\delta_{ij} \ , \tag{1}$$

where $\hat{r}_i$ is the radial unit vector. Find the coefficient functions $A(r), B(r), C(r)$ in two ways:

1. *By solving the linearized Einstein equation $\Box h_{\mu\nu} = -\kappa(T_{\mu\nu} - \frac{1}{2}T\eta_{\mu\nu})$ in de Donder gauge.*
2. *By taking the linearized limit of the Schwarzschild solution.*

If the two solutions are different, find the gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}$ that relates them.

Question 3. *Now, repeat Question 1, but for a massless photon moving in the linearized geometry of Question 2. Namely, find the photon's deflection angle by integrating the x component of the geodesic equation. This was the first observational test of GR!*

Part II: Falling into the light

Question 4. *In flat spacetime, write down the stress-energy tensor $T^{\mu\nu}$ describing a photon with energy E that passes through the origin, moving in the positive z direction. Do this in two coordinate systems:*

1. *Cartesian coordinates* $x^\mu = (t, x, y, z)$.

2. *Lightcone coordinates* $x^\mu = (u, v, x, y)$*, with* $u \equiv (t+z)/\sqrt{2}$ *and* $v \equiv (t-z)/\sqrt{2}$.

Question 5. *Now, let us write curved geometries using lightcone coordinates* $x^\mu = (u, v, x, y)$ *and a lightcone basis* $I = (\hat{u}, \hat{v}, \hat{x}, \hat{y})$ *for the internal space, so the non-vanishing components of* η_{IJ} *read:*

$$\eta_{\hat{u}\hat{v}} = \eta_{\hat{v}\hat{u}} = -1 \ ; \quad \eta_{\hat{x}\hat{x}} = \eta_{\hat{y}\hat{y}} = 1 \ . \tag{2}$$

In this setup, consider a class of vielbeins whose non-vanishing components read:

$$e_u^{\hat{u}} = e_v^{\hat{v}} = e_x^{\hat{x}} = e_y^{\hat{y}} = 1 \ ; \quad e_v^{\hat{u}} = A(v, x, y) \ . \tag{3}$$

Find the stress-energy tensor T_{IJ} *that generates the geometry* (3)*. Find the function* $A(v, x, y)$ *that corresponds to the photon from Question 4.*

Question 6. *Now, consider a probe photon passing by, flying initially in the* <u>*negative*</u> z *direction. As in Question 1, assume an impact parameter* b *along the* x *axis, and find the deflection angle due to the gravitational field of the photon from Question 5. What will happen to a probe photon that moves in the* <u>*positive*</u> z *direction?*