GENERAL RELATIVITY MIDTERM EXAM

Exercise 1. *Consider the unit sphere, with coordinates $x^\mu = (\theta, \phi)$. The internal-space indices take values $I = \hat{\theta}, \hat{\phi}$. The internal metric is the Euclidean δ_{IJ}. The non-vanishing components of the vielbein are:*

$$e_\theta^{\hat{\theta}} = 1 \ ; \quad e_\phi^{\hat{\phi}} = \sin\theta \ . \tag{1}$$

1. *Write out in components the compatibility condition $\partial_{[\mu} e^I_{\nu]} + \omega^{IJ}_{[\mu} e_{\nu]J} = 0$. Solve it for the components of the spin-connection ω^{IJ}_μ.*

2. *Consider a circle $\theta = const.$ Express the length element ds in terms of $d\phi$. Write the components of the unit tangent u^I. Find the curvature Du^I/ds. For what values of θ is this circle a geodesic? For what values of θ does Du^I/ds become infinite? Why?*

Exercise 2. *Consider an exponentially expanding universe (also known as de Sitter space), with coordinates $x^\mu = (t, x, y, z)$. The internal-space indices take values $I = \hat{t}, \hat{x}, \hat{y}, \hat{z}$. The internal metric is the Minkowski η_{IJ}. The non-vanishing components of the vielbein are:*

$$e_t^{\hat{t}} = 1 \ ; \quad e_x^{\hat{x}} = e_y^{\hat{y}} = e_z^{\hat{z}} = e^{Ht} \ . \tag{2}$$

Here, H is a constant, known as the Hubble expansion rate (and the final "e" is an exponent).

1. *The spin-connection ω^{IJ}_μ has different types of components, up to permutations of the spatial axes (x, y, z). These are $\omega_t^{\hat{t}\hat{x}}, \omega_t^{\hat{x}\hat{y}}, \omega_x^{\hat{t}\hat{x}}, \omega_x^{\hat{t}\hat{y}}, \omega_x^{\hat{x}\hat{y}}, \omega_x^{\hat{y}\hat{z}}$. The symmetry of spatial reflections implies that all of these must vanish, except one. Which one?*

2. *Find the non-vanishing components of ω^{IJ}_μ by solving the compatibility condition $\partial_{[\mu} e^I_{\nu]} + \omega^{IJ}_{[\mu} e_{\nu]J} = 0$.*

3. *A free particle of mass m is moving in the spacetime (2), in the x direction (i.e. in the tx plane). Spatial translation symmetry implies that the momentum p_x is conserved. Express the conserved momentum p_x in terms of the coordinate velocity dx/dt.*

4. *What value of dx/dt corresponds to the speed of light (i.e. makes the proper time $d\tau$ vanish, or the momentum become infinite)?*

5. *Starting at $(t, x) = (t_0, 0)$, what is the farthest spatial position x that the particle can reach?*