Quantum Cryptanalysis

Manuel Goulão

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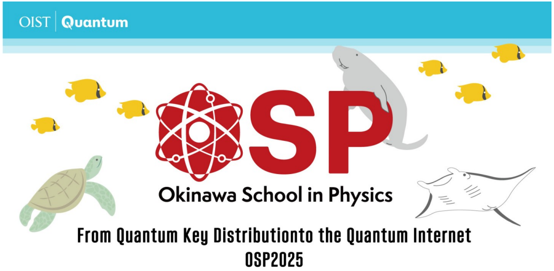


Overview

Contents

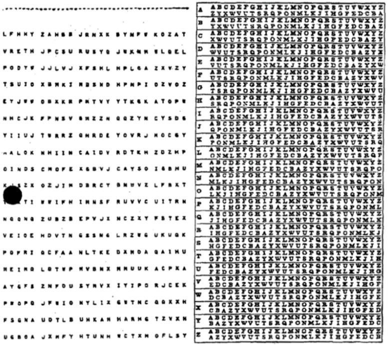
Cryptography & Cryptanalysis

How to communicate securely?

- Unconditional security (e.g., OTP)
- Computational security (e.g., RSA)

Cryptanalysis

- By hand
- Early automata
- Classical computers
- Quantum computers



NSA One-Time Pad (Source: Wikimedia)

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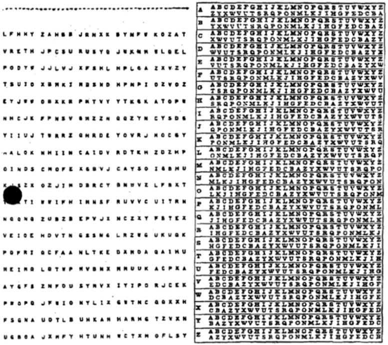

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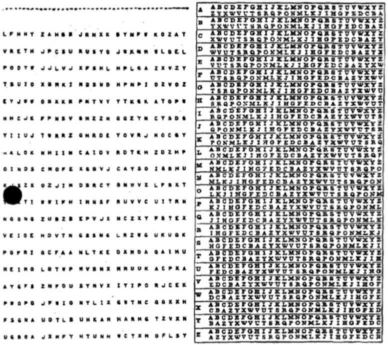

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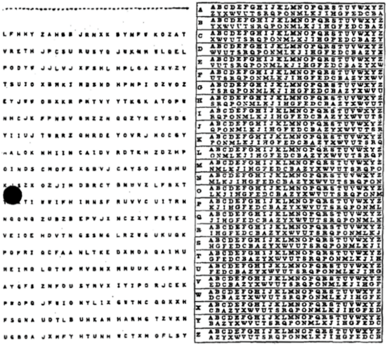

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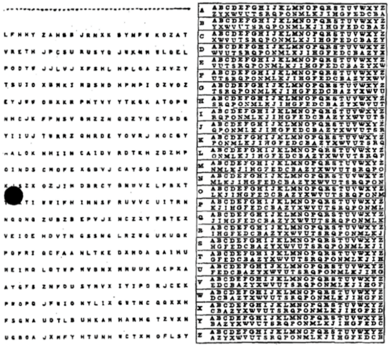

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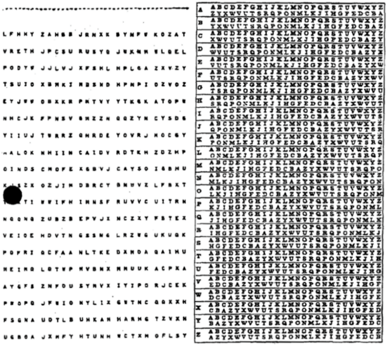

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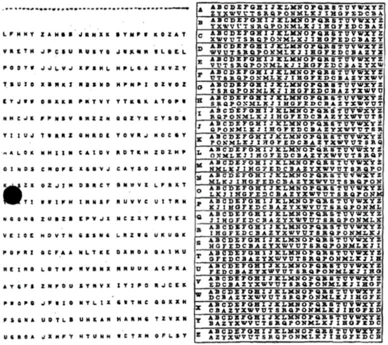

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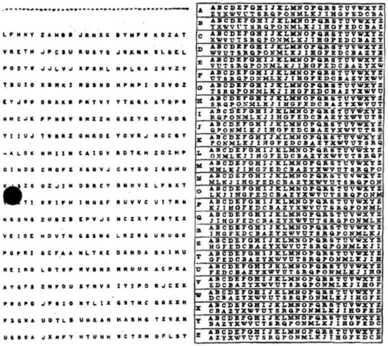

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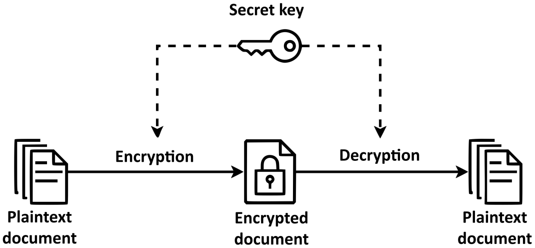
Symmetric Cryptosystems

- 1 shared secret key
- AES, SHA, etc.
- *Computational assumptions:* highly unstructured/nonlinear problems



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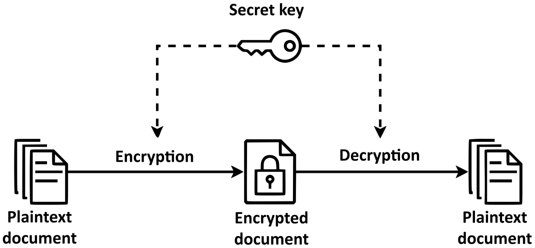

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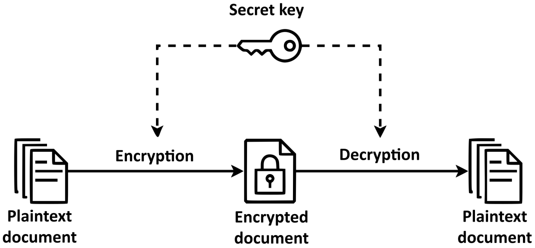

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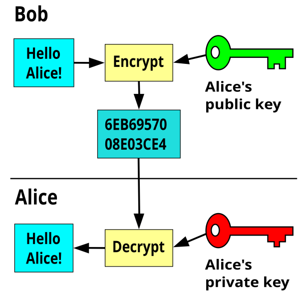

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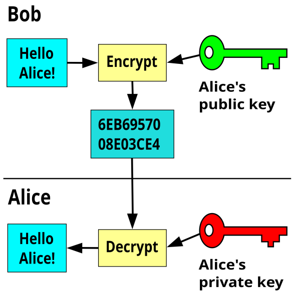

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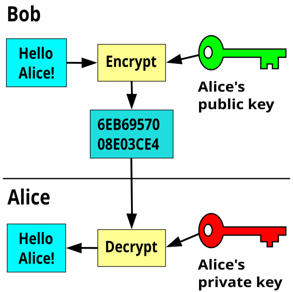

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Current Deployment

- PKC is ubiquitous in the information-world (internet, credit cards, messaging, etc.)
- *Harvest/Store Now Decrypt Later* (HNDL/SNDL)
- Push to standardize post-quantum — already in TLS (hybrid)

RSA (factoring)

PK: $1 < e < \phi(p \cdot q)$;
SK: $d \equiv e^{-1} \pmod{\phi(p \cdot q)}$

Encryption: $c \equiv m^e \pmod{p \cdot q}$

Decryption: $m \equiv c^d \pmod{p \cdot q}$

El Gamal (dlog)

SK: $x \in \mathbb{Z}_q$, **PK**: $h = g^x$

Encryption: $c_1 = g^k$, $c_2 = m \cdot h^k$

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Security of post-quantum schemes

- **NIST Standard** (2022-25):
  - Kyber (Lattice, KEM)
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  - Dilithium, FALCON (Lattice, DS)
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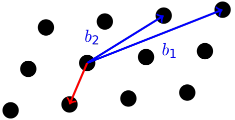

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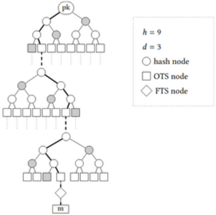

Source: SPHINCS+

- Not thoroughly studied classically...
- ... even less for **quantum attacks**.

Source: Sendrier SP'17

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Breaking Rainbow Takes a Weekend on a Laptop

Ward Beullens

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Abstract. This work introduces new key recovery attacks against the Rainbow signature scheme, which is one of the three finalist signature schemes still in the NIST Post-Quantum Cryptography standardization project. The new attacks outperform previously known attacks for all the parameter sets submitted to NIST and make a key-recovery practical for the SL 1 parameters. Concretely, given a Rainbow public key for the SL 1 parameters of the second-round submission, our attack returns the corresponding secret key after on average 53 hours (one weekend) of computation time on a standard laptop.

An efficient key recovery attack on SIDH

Wouter Castryck[1,2] and Thomas Decru[1]

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Broken NIST-PQC finalists

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1. All parties are classical — computation is PPT:
   classical security
2. Adversary has a quantum computer — computation is QPT:
   post-quantum security
3. **Communication** is quantum (ptx/ctx/keys are still classical):
   quantum security (qCPA/qCCA/qCMA)

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Grover's search

INPUT: $f : \{0, \dots, N-1\} \to \{0, 1\}$ where $f(x) = 1$ for a single x

OUTPUT: ω such that $f(\omega) = 1$ with high probability

- Unstructured search (db, key, etc.)
- **Classical solution:** $\mathcal{O}(N)$
- **Quantum solution:** $\mathcal{O}(\sqrt{N})$
- **Intuition:** divide security level by 2

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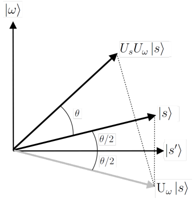

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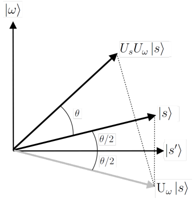

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$$U_\omega = I - 2\,|\omega\rangle\langle\omega|$$

$$U_s = 2\,|s\rangle\langle s| - I$$

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Impact to cryptography

- **Symmetric-key encryption:** AES-128, AES-192, AES-256
  - brute-force key: $256 \rightarrow 128$,
    $192 \rightarrow 96$,
    $128 \rightarrow 64$
- **Cryptographic hash function:** SHA-256, SHA3-384, SHA3-512
  - preimage-resistence: $512 \rightarrow 256$,
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  - collision-resistance: $512 \rightarrow 170$,
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BHT algorithm (Brassard, Høyer, Tapp, 1998):

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- Ekerå and Gidney (2021) $\sim$**20 million qubits**, 8 hours to break RSA-2048
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- Massive impact to **public-key cryptography standards** (in use even today!)
- Changed the field of **secure communication** and **quantum computing**
- Ekerå and Gidney (2021) $\sim$**20 million qubits**, 8 hours to break RSA-2048
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Develop *quantum-safe cryptography*

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$$\texttt{INPUT:} \begin{cases} N = p \cdot q \\ f(x) = a^x \pmod N \end{cases}$$

1. Start with uniform superposition $|x\rangle = \frac{1}{2^{2n}} \sum_{s=0}^{2^{2n}-1} |s\rangle$ $(\mathcal{O}(n))$
2. Evaluate f on $|x\rangle$ $(\mathcal{O}(n^3))$
3. Preform Quantum Fourier Transform QFT$(f(|x\rangle))$ $(\mathcal{O}(n^2))$
4. Measure to read output, the period of f $(\mathcal{O}(n))$

Find period $\overset{\text{classical}}{\Longrightarrow}$ **Factor** N

- Classical complexity (GNFS): $\mathcal{O}(e^{1.9(n)^{1/3}\log(n)^{2/3}})$
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Classical reduction

$$\texttt{INPUT:} \begin{cases} a \\ r, \text{ order of } a \quad (\text{smallest } i,\ a^i = 1 \pmod N) \\ \quad r \text{ is even, } a^{r/2} \neq -1 \pmod N \end{cases}$$

1. **if** $k = \gcd(a, N) \neq 1$, **return** $(k, N/k)$
2. $a^r = 1 \pmod N$, so $N \mid (a^r - 1)$
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4. But $N \nmid (a^{r/2} - 1)$ and $N \nmid (a^{r/2} + 1)$ $(a^{r/2} \neq \pm 1)$
5. **return** $\left(\gcd(a^{r/2} + 1, N), \gcd(a^{r/2} - 1, N)\right)$

Example

$$N = 15,\ a = 7,\ r = 4$$

$$\begin{cases} r \text{ is even} \\ 7^{4/2} = 4 \neq -1 \pmod{15} \end{cases}$$

$$p = \gcd(7^2 - 1, 15) = 3$$
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Quantum Order Finding

INPUT: $\begin{cases} N = p \cdot q \\ U : |x\rangle |0\rangle \to |x\rangle |a^x \pmod N\rangle \end{cases}$

0. $t \propto \lceil \log_2 N^2 \rceil$, $n = \lceil \log_2 N \rceil$
1. $|\psi_1\rangle = (H^{\otimes t} \otimes I^{\otimes n}) |0\rangle^{\otimes t} |0\rangle^{\otimes n} = \sum_{x=0}^{2^t-1} |x\rangle |0\rangle$
2. $|\psi_2\rangle = U |\psi_2\rangle = \sum_{x=0}^{2^t-1} |x\rangle |a^x \pmod N\rangle$
3. Measure $y_0 = a^{x_0} \pmod N$
   $|\psi_3\rangle = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |x_0 + kr\rangle$
4. $|\psi_4\rangle = \text{QFT}_{2^t} |\psi_4\rangle =$
   $\frac{1}{\sqrt{m}} \frac{1}{\sqrt{2^t}} \sum_{y=0}^{2^t-1} \sum_{k=0}^{m-1} e^{2\pi i \frac{(x_0+kr)y}{2^t}} |y\rangle$
5. Measure $y \approx s\frac{2^t}{r}$ $(s \in \mathbb{Z})$
   $\frac{s}{r}$ is a convergent of $\frac{y}{2^t} \Longrightarrow$ Recover r

Example

$N = 15,\ a = 7,\ t = 8$

$|\psi_1\rangle = \frac{1}{\sqrt{256}} \sum_{x=0}^{255} |x\rangle |0\rangle$

$|\psi_2\rangle = \frac{1}{\sqrt{256}} \sum_x |x\rangle |7^x \pmod{15}\rangle$

$|\psi_3\rangle = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |x_0 + 4k\rangle$

$\sum_k e^{2\pi i \frac{4k}{2^t} y}$ peaks at $y \approx s\frac{2^t}{4}$

Measure $y = 64$

$64/256 = 1/4 \Longrightarrow r = 4$

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$N = 15$, $a = 7$, $t = 8$

$|\psi_1\rangle = \frac{1}{\sqrt{256}} \sum_{x=0}^{255} |x\rangle |0\rangle$

$|\psi_2\rangle = \frac{1}{\sqrt{256}} \sum_x |x\rangle |7^x \pmod{15}\rangle$

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$\sum_k e^{2\pi i \frac{4k}{2^t} y}$ peaks at $y \approx s\frac{2^t}{4}$

Measure $y = 64$

$64/256 = 1/4 \implies r = 4$

Shor's Algorithm

Quantum Order Finding

INPUT: $\begin{cases} N = p \cdot q \\ U : |x\rangle|0\rangle \to |x\rangle|a^x \pmod N\rangle \end{cases}$

0. $t \propto \lceil \log_2 N^2 \rceil$, $n = \lceil \log_2 N \rceil$
1. $|\psi_1\rangle = (H^{\otimes t} \otimes I^{\otimes n})|0\rangle^{\otimes t}|0\rangle^{\otimes n} = \sum_{x=0}^{2^t-1}|x\rangle|0\rangle$
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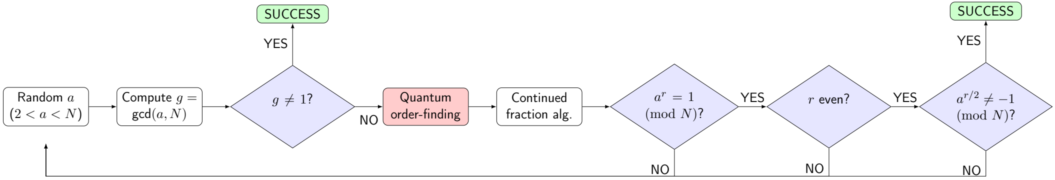

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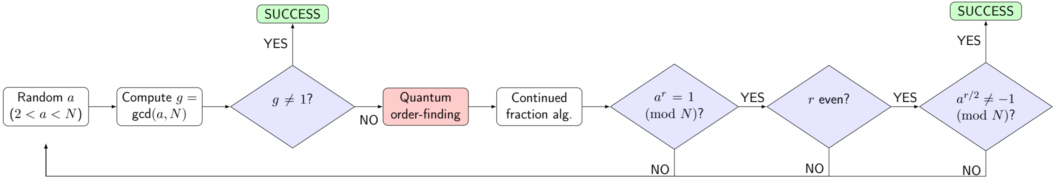

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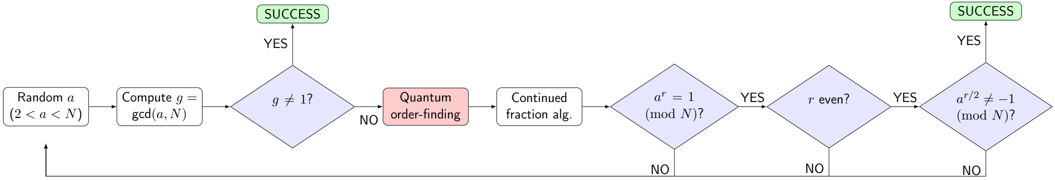

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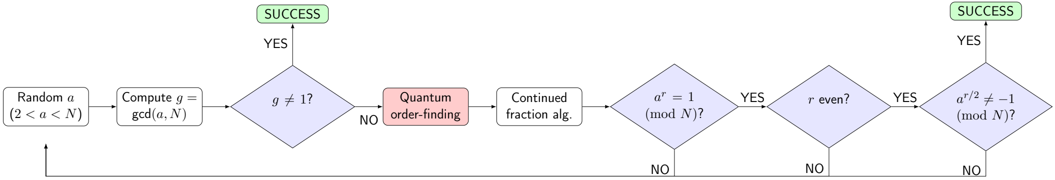

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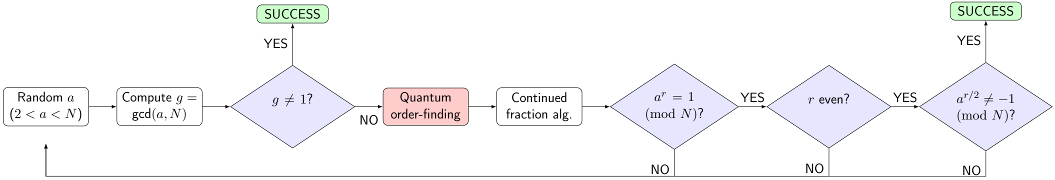

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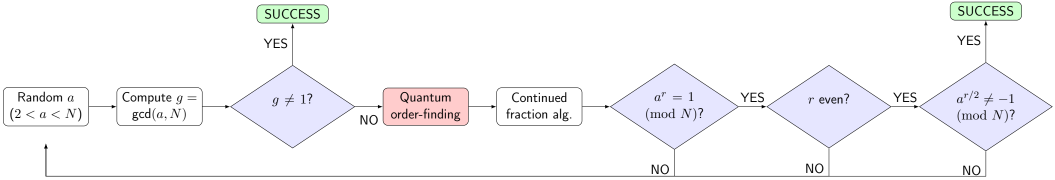

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Hidden Subgroup Problem

Intuition

Find *hidden periodic* structure in group

- Given group G and function $f : G \to X$
- f is constant on cosets of unknown subgroup $H \subseteq G$
- **Goal:** Determine subgroup H

Connection to Shor's algorithm

- $G = \mathbb{Z}_{N^2}$ (integers modulo N^2)
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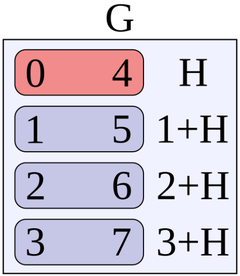

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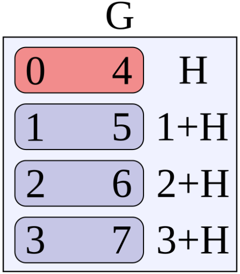

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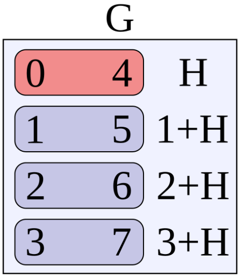

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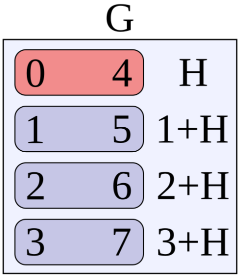

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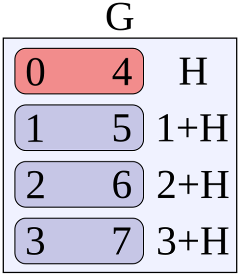

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Hidden Subgroup Problem

Problem	Quantum algorithm	Abelian?	Polynomial time?
Deutsch's problem	Deutsch's/Deutsch-Jozsa algorithm	Yes	Yes
Simon's problem	Simon's algorithm	Yes	Yes
Order finding	Shor's order finding algorithm	Yes	Yes
Discrete logarithm	Shor's algorithm for discrete logarithms	Yes	Yes
Period finding	Shor's algorithm	Yes	Yes
Abelian stabilizer	Kitaev's algorithm	Yes	Yes
Graph Isomorphism	None	No	No
Shortest vector problem	None	No	No

List of HSP quantum algorithms. (Source: Wikipedia)

Contents

Reversible Computation

- Classical computation often erases information — **irreversible**
- Quantum evolution is unitary (also, no fanout) — **reversible**
- **Compile** any irreversible $f : \mathbb{Z}_2^n \to \mathbb{Z}_2^m$ into a reversible $\overline{f} : \mathbb{Z}_2^{n+m} \to \mathbb{Z}_2^{n+m}$

$$\overline{f}((x, y)) = (x, y \oplus f(x))$$

Example — AND

$$\mathsf{AND}(a, b) = a \wedge b$$

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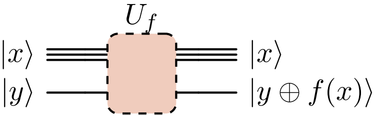

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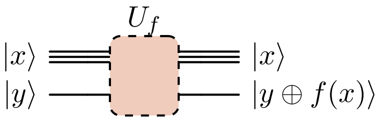

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Output register $|y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ may also be in a superposition

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$$= \sum_i (-1)^{f(x_i)} \alpha_i |x_i\rangle |y\rangle$$ ($|y\rangle$ is separable, often ommited)

Exploit interference pattern of f controlled by x_i

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Simon's problem

Given: $f_{\mathbf{s}} : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$

Promise: $f_{\mathbf{s}}(\mathbf{x}) = f_{\mathbf{s}}(\mathbf{x} \oplus \mathbf{s})$

Goal: $\mathbf{s} \in \mathbb{Z}_2^n$

Classical solution:

let $L = \{\}$
for $\mathbf{x}_i \in \mathbb{Z}_2^n$
  let $\mathbf{y}_i = f(\mathbf{x}_i)$
  if $(\mathbf{x}_j, \mathbf{y}_j) \in L$ st $\mathbf{y}_i = \mathbf{y}_j$
    return $\mathbf{s} = \mathbf{x}_i \oplus \mathbf{x}_j$
  else
    append $(\mathbf{x}_i, \mathbf{y}_i)$ to L
return $\mathbf{s} = 0$

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Promise: $f_{\mathbf{s}}(\mathbf{x}) = f_{\mathbf{s}}(\mathbf{x} \oplus \mathbf{s})$

Goal: $\mathbf{s} \in \mathbb{Z}_2^n$

Classical solution:

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Classical: 2^n queries

Quantum: αn queries

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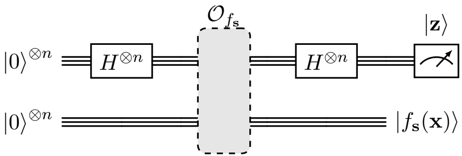
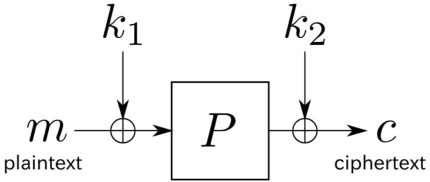


Superposition Attacks

Block Ciphers

1. First proposal [KM10] — 3-round Feistel distinguisher
2. Even-Mansour key-recovery [KM12]:



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$$\mathsf{Enc}_{k_1,k_2}(m) = P(m \oplus k_1) \oplus k_2$$

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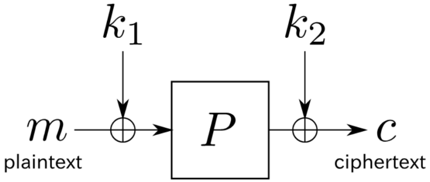

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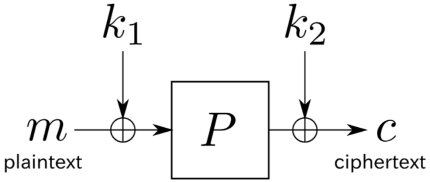
1. First proposal [KM10] — 3-round Feistel distinguisher

2. Even-Mansour key-recovery [KM12]:



The EM cipher

[KM12]

$$\mathsf{Enc}_{k_1,k_2}(m) = P(m \oplus k_1) \oplus k_2$$

$$f_{k_1}(m) = \mathsf{Enc}_{k_1,k_2}(m) \oplus P(m)$$

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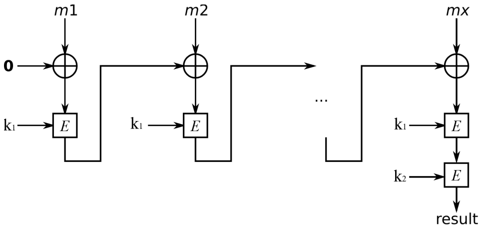
Simon's algorithm to recover k_1

$$k_2 = \mathsf{Enc}_{k_1,k_2}(m) \oplus P(m \oplus k_1)$$

Superposition Attacks

MACs

- Forgery attack — CBC-MAC [KLLN16]
- *More:* LightMAC, PolyMAC, GCM-SIV2, Poly1305, ... [BLNS21]



Source: wikipedia

$$\text{CBCMAC}(m_1|m_2) = E_{k_2}\left(E_{k_1}\left(m_2 \oplus E_{k_1}\left(m_1\right)\right)\right)$$

$$f : \mathbb{Z}_2 \times \mathbb{Z}_2^n \to \mathbb{Z}_2^n$$
$$b, x \to \text{CBCMAC}(m_b|x)$$

$$f(b,x) = E_{k_2}(E_{k_1}(x \oplus E_{k_1}(m_b)))$$

$$f(b,x) = f(b \oplus 1, x \oplus E_{k_1}(m_0) \oplus E_{k_1}(m_1))$$

Simon's: $\Rightarrow E_{k_1}(m_0) \oplus E_{k_1}(m_1)$

1. Pick x
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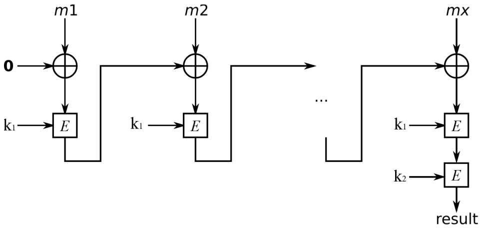

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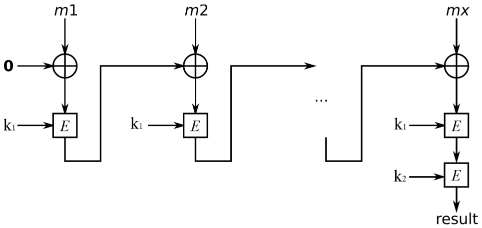

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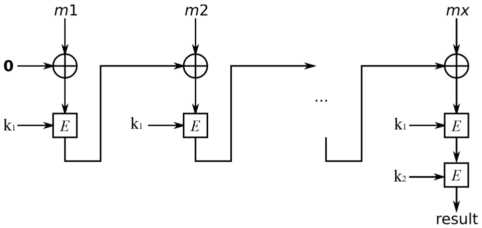

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- Are the new post-quantum proposals also **Q2-secure**?
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