Qudits and high-dimensional entanglement

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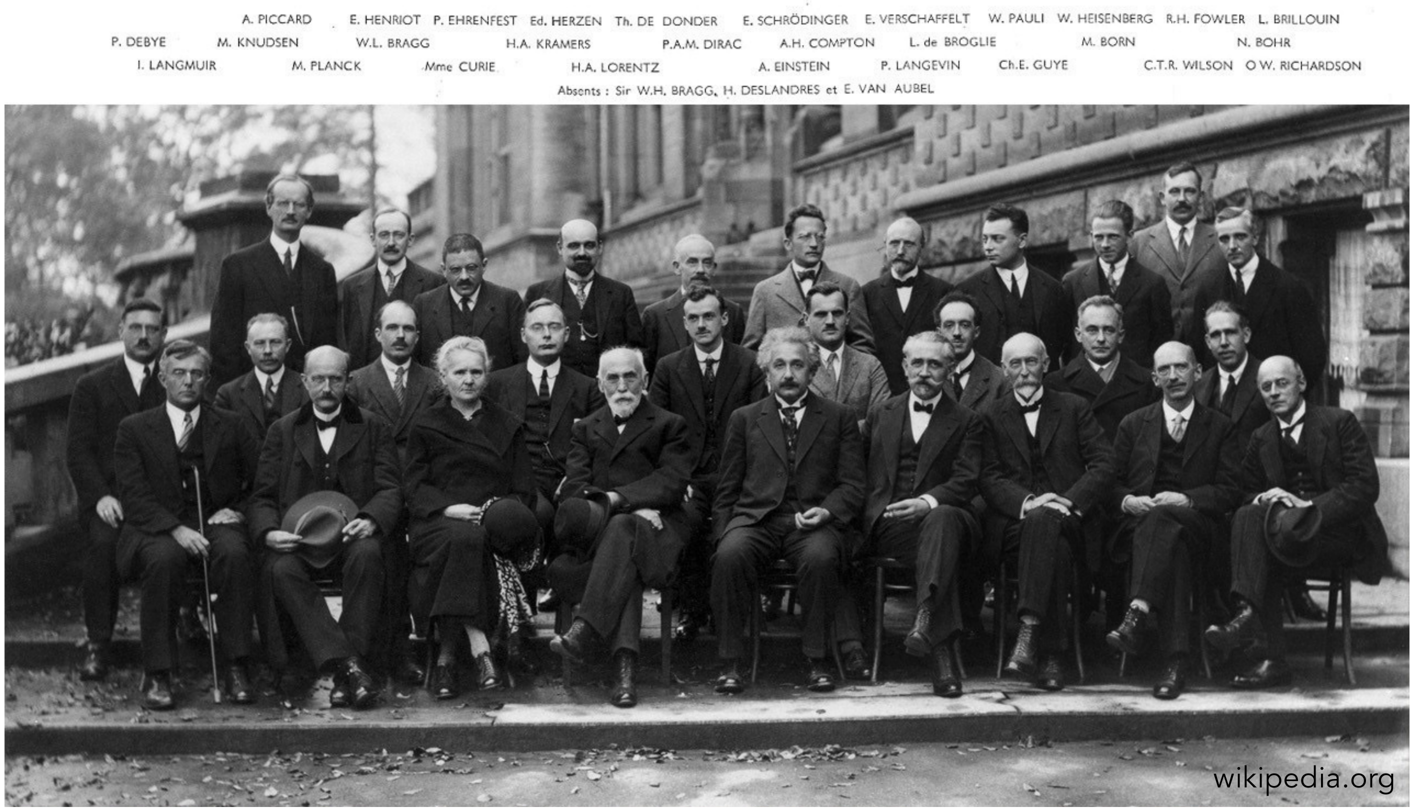
INTERNATIONAL YEAR OF
Quantum Science
and Technology

Physicists then



Physicists now

running since 1927…

http://thetenthwatch.com/

Qudits@UQ

THE UNIVERSITY OF QUEENSLAND
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www.quantum-atbp.org

Welcome!

Quantum Atbp [at-i-ba-pa]—Quantum and others—in my native Filipino language is the home of **Qudits@UQ.**

We are an experimental quantum physics group at the School of Mathematics and Physics of the University of Queensland. We are interested in all things qudits—quantum systems with more than two levels. Our aim is to discover new physics and develop quantum


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Interest over time
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75
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Note
Note
Average
1 Jan 2004
1 June 2011
1 Nov 2018

My interests…

Quantum foundations: entanglement, nonlocality, "non-classicality"

High-dimensional quantum information (the discrete variable kind)

Photonic quantum technologies

Science communication

Science diplomacy

UNESCO

ISO

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INTERNATIONAL YEAR OF
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Information is physical—information is carried by physical systems.

Different physical systems have different rules.

Plan…

Introduce dits and qudits

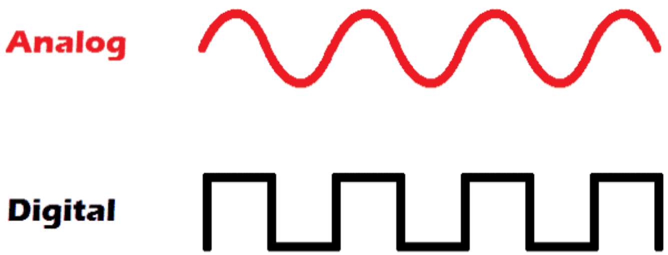
Communications tasks: how are dits and qudits similar, and different

Introduce entanglement

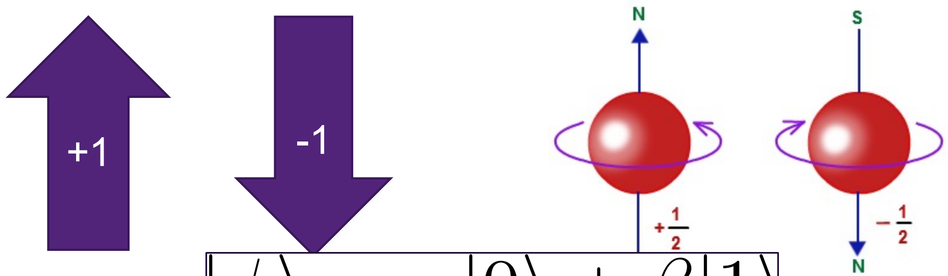
Introduce high-dimensional entanglement

Some remarks on entanglement and nonseparability

The Bit



The simplest classical system is the two-state system–the **bit**.

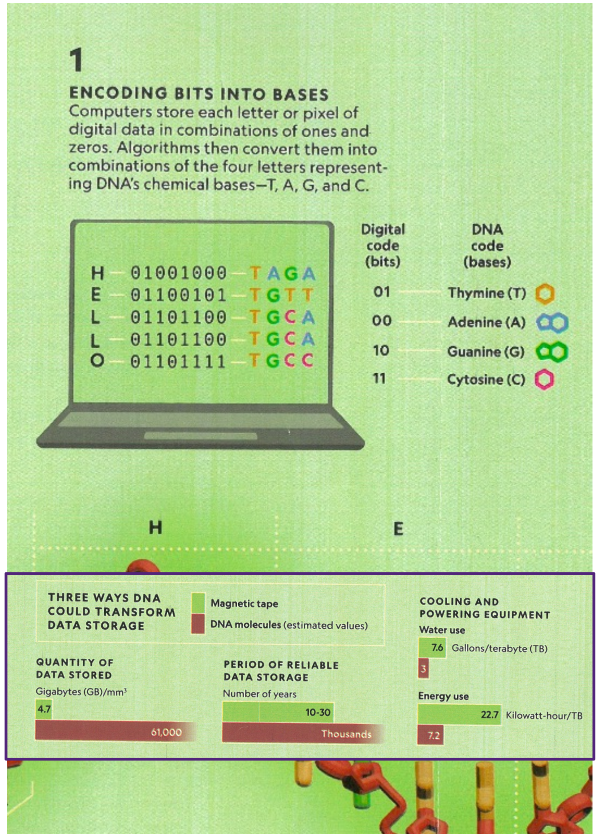


qubits

$$\boxed{|\psi\rangle = \alpha|0\rangle + \beta|1\rangle}$$

electricalfundablog.com, moaid.codes, spectator.com.au, quantum-field-theory.net

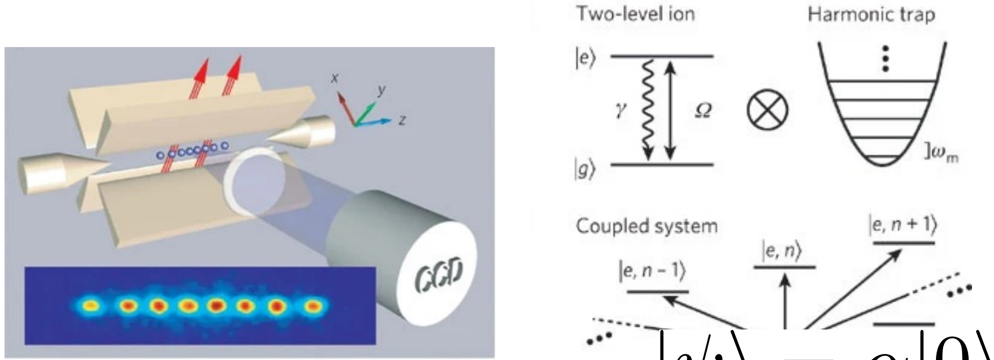
The Dit



National Geographic

Dit is a classical system that has *d*-possible states.

qudits



Blatt and Wineland, Nature 2008

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \ldots + \gamma|d-1\rangle$$

Example: Trapped ion qudit

npj | quantum information

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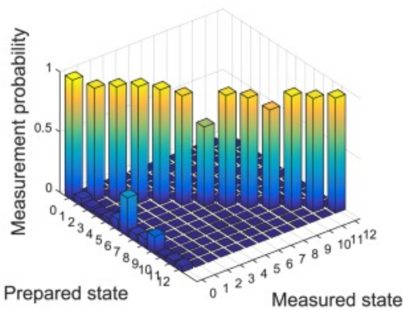
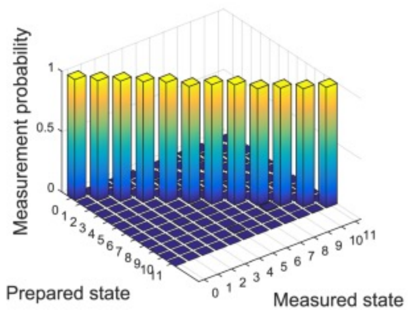
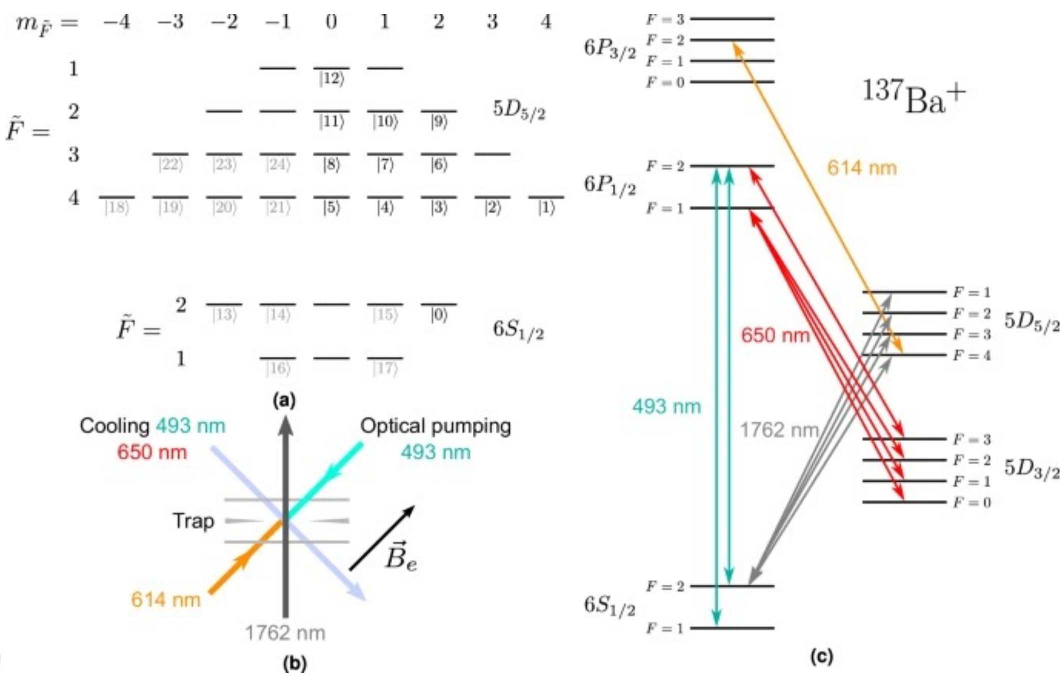
Article | Open access | Published: 24 May 2025

Control and readout of a 13-level trapped ion qudit

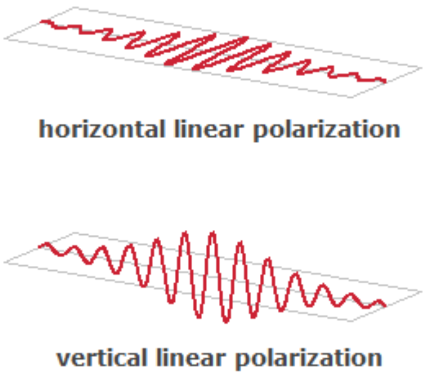
Pei Jiang Low, Brendan White & Crystal Senko



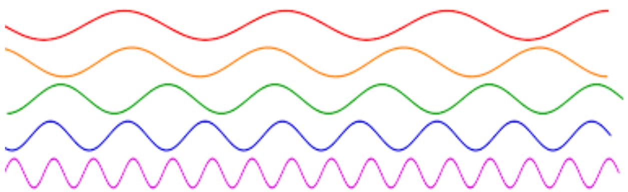




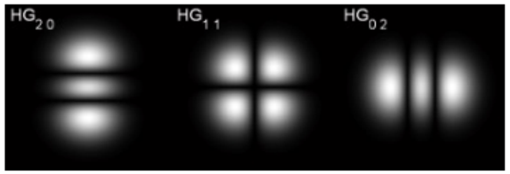
How to make qubits/qudits with light?



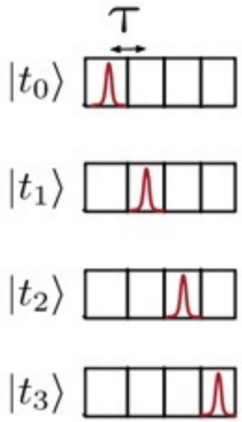
polarisation



wavelength/frequency



transverse mode/shape
(OAM/twisted light)

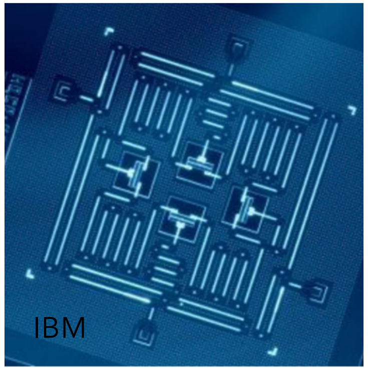


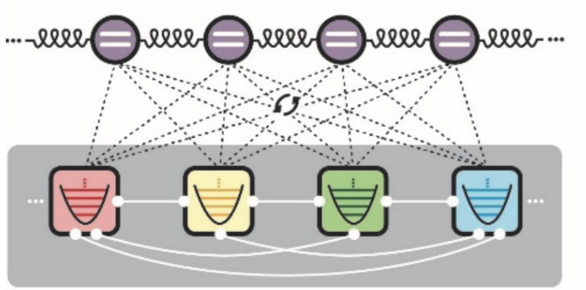
time bins

rp-photonics.com

Aside: matter-based vs photonic quantum computer

matter

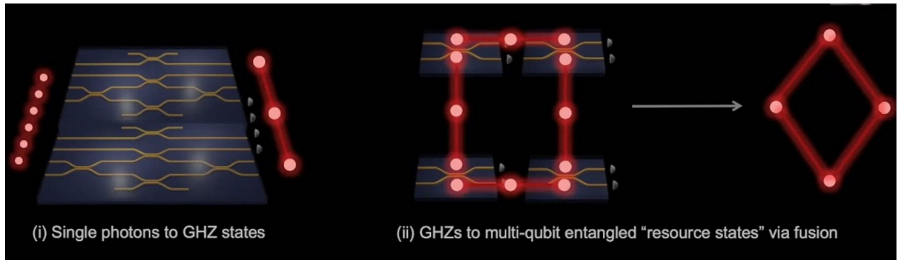




- Examples: spin, superconducting, ions, neutral atoms
- Qubits are fixed in place
- Time-dependent controls are switched on and off in a particular order
- Closer to conventional silicon computer

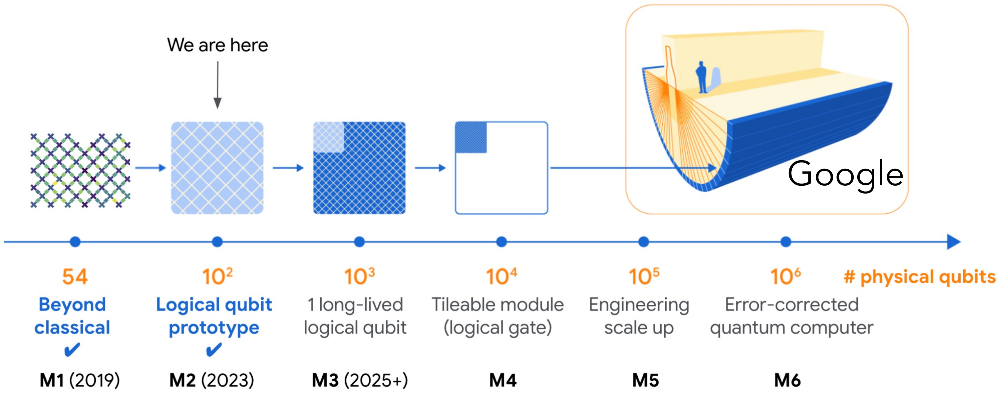
photonic

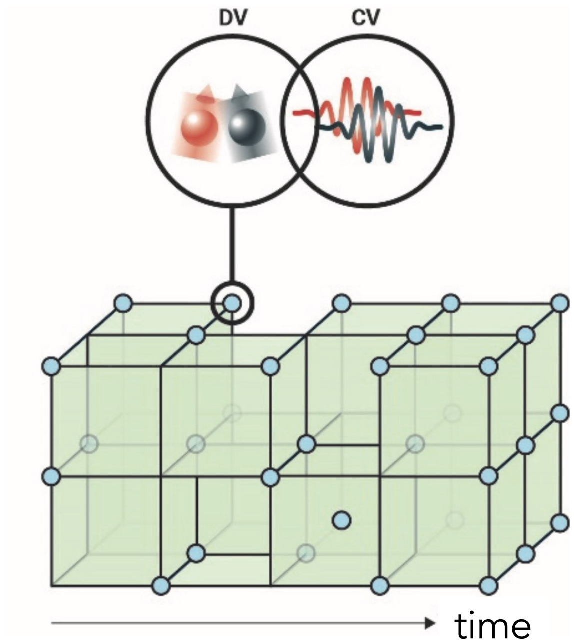
PsiQuantum



- Example: photons, electromagnetic field
- Pulses of light moving through fibres or waveguides
- Entangled qubits are continuously created and measured.
- Closer to an all-optical network.

Aside: matter is different from photonic





For matter: the physical qubits "live" on the chip.

For photonic: the physical qubits are generated and consumed all the time.

Using photonic qudits (for quantum communication, quantum information in general) is a double-edged sword

Metric	Qubits	Qudits	
Information capacity	1 bit per photon	$\log_2 d$ bits per photon	✅
Noise tolerance/ QBER threshold	11% (BB84)	18-20% for d=4	✅
Bell inequality violation	CHSH-type	CGLMP-type, stronger violations for better for device-independent QKD	✅
Detector requirements	2 detectors per basis	d detectors per basis	❌
Loss resilience	1 photon lost= 1 bit lost	1 photon lost=$\log_2 d$ bits lost	❌

Erhard et al, Light Sci Appl 7 17146(2018)

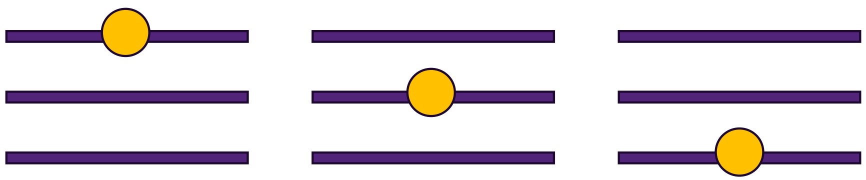
Using photonic qudits (for quantum communication, quantum information in general) is a double-edged sword

Metric	Qubits	Qudits	
Hardware complexity	mature, relatively simple	multi-path interferometers, mode sorters, which can be more complex and lossy	🤓
Experimental maturity	Commercial QKD systems available, deployed worldwide	lab and field demonstrations (e.g. 145 km fibre for ququart, *d=4*, *Vagniluca et al., Phys. Rev. Applied 14, 014051 2020*)	🤓
Integration potential	already integrated (silicon and TFLN platforms)	relatively new, feasible for moderate values of *d*	🤓
Use in networks and repeaters	well-studied	not well-studied	🤓

Dits and qudits

d=number of simultaneously distinguishable states

Dits (trit, *d=3*)



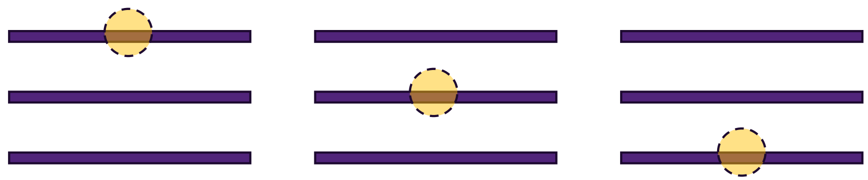
0 $\quad$ 1 $\quad$ 2

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\0\\1\end{pmatrix}$$

We have a "pure state" when we can specify the vector with certainty.

Pure states are specified by vectors.

Qudits (qutrit, *d=3*)



$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle + \gamma\,|2\rangle$$

complex coefficients

$$|\psi\rangle = \alpha\begin{pmatrix}1\\0\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\\0\end{pmatrix} + \gamma\begin{pmatrix}0\\0\\1\end{pmatrix}$$

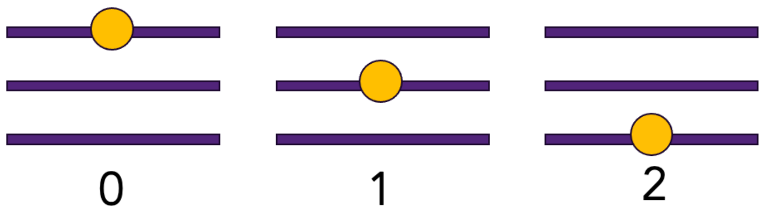
Probability (modulus squared) $\quad |\alpha|^2 \quad |\beta|^2 \quad |\gamma|^2$

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$$

Dits and qudits can also be mixtures

After fixing an orthonormal basis, we can write the density matrix:

Dits (trit, *d*=3)



$$c = \begin{bmatrix} t_0 & 0 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & t_2 \end{bmatrix}$$

Probabilities: $t_0 \quad t_1 \quad t_2$

$$t_0 + t_1 + t_2 = 1$$

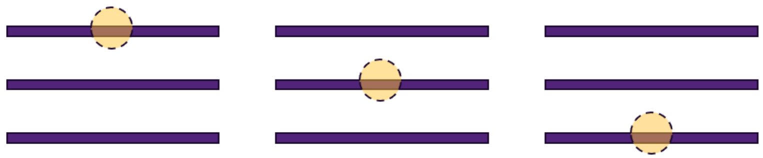
The density matrix is Hermitean, non-negative, and its trace is 1.

For dits, the off-diagonal elements are 0.

The dit space is in the qudit space

The density matrix of a qudit in a pure state is: $\rho = |\psi\rangle\langle\psi|$

Qudits (qutrit, *d*=3)



$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle + \gamma\,|2\rangle$$

$$\rho = \begin{bmatrix} |\alpha|^2 & \alpha\beta^* & \alpha\gamma^* \\ \beta\alpha^* & |\beta|^2 & \beta\gamma^* \\ \gamma\alpha^* & \gamma\beta^* & |\gamma|^2 \end{bmatrix} \qquad c = \begin{bmatrix} t_0 & 0 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & t_2 \end{bmatrix}$$

Qudits have off-diagonal elements (coherence).

The dit corresponds to the restriction to use pure states in some fixed orthonormal basis and their mixtures.

The density matrix (operator) for qudits is given by:

$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n| \qquad \sum_n p_n = 1$$

pure state if: $p_n = \delta_{mn}$

Exercise 1 (5-ish minutes):

True or False: The density operator shown is a maximally mixed state.

Plan…

Introduce dits and qudits

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Some remarks on entanglement and nonseparability

Qudit measurements

Projective measurements– reading one dit from a qudit

Holevo bound- one qudit can encode at most one dit of information.

$\mathrm{P}(j) = |\varphi_j\rangle \langle\varphi_j|$ orthonormal $\quad \mathrm{Prob}(j) = \mathrm{tr}[\rho \mathrm{P}(j)]$

The number of projective measurements cannot exceed *d*, *j<d*.

More generally…

Assign to every outcome j, a positive operator *M(j)*, j can be greater than d

$$P(j) = \mathrm{Tr}(\rho \mathrm{M}(j))$$

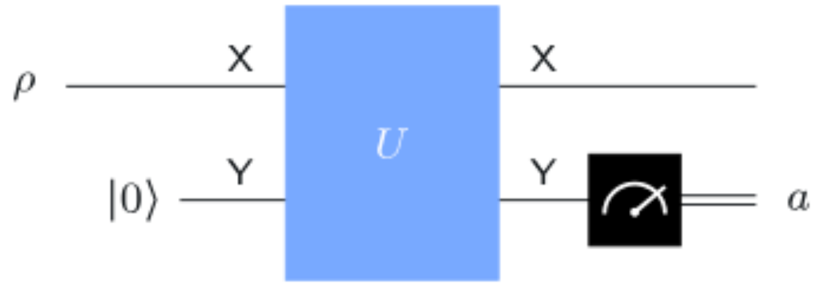
The set of operators M form a POVM. M(j) is Hermitean and positive, and $\sum_j \mathrm{M}(j) = \mathbb{I}$

Some elements in M can be non-commuting (incompatible).

Aside: Naimark's (dilation) theorem

$$\mathrm{tr}[\rho \mathrm{M}(j)] = \mathrm{tr}\left[U\left(\rho \otimes \rho_0\right)U^*\mathrm{P}(j)\right]$$

The outcome of a POVM can be simulated with a projective measurement in a *suitable higher-dimensional space*.



quantum.cloud.ibm.com

Communciation tasks with dits and qudits

1. Minimum error communication

2. Communication of uniform partial ignorance

3. Random sequences with non-overlaps

(I'll use the board now…)

Heinosaari and Hillery, Contemporary Physics 65 (2024)

Communciation taks with dits and qudits: summary

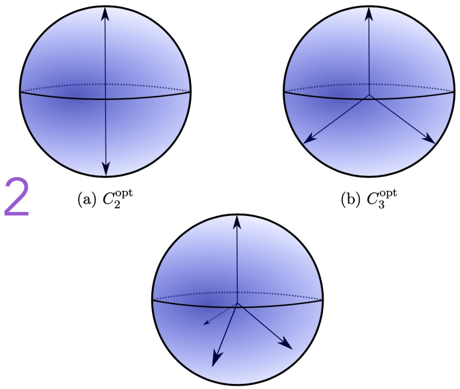
1. Minimum error communication   no qubit/qudit advantage

2. Communication of uniform partial ignorance   not possible with a bit/dit

   The minimum classical and quantum dimension for some communication tasks differ.

   The minimum qudit dimension for some tasks is greater than 2 (tasks not possible with a qubit).

   There are some interesting open questions.



(a) C_2^{opt} (b) C_3^{opt} (c) C_4^{opt}

3. Random sequences with non-overlaps   can be applied to authentication

Heinosaari and Hillery, Contemporary Physics 65 (2024)

Plan…

Introduce dits and qudits

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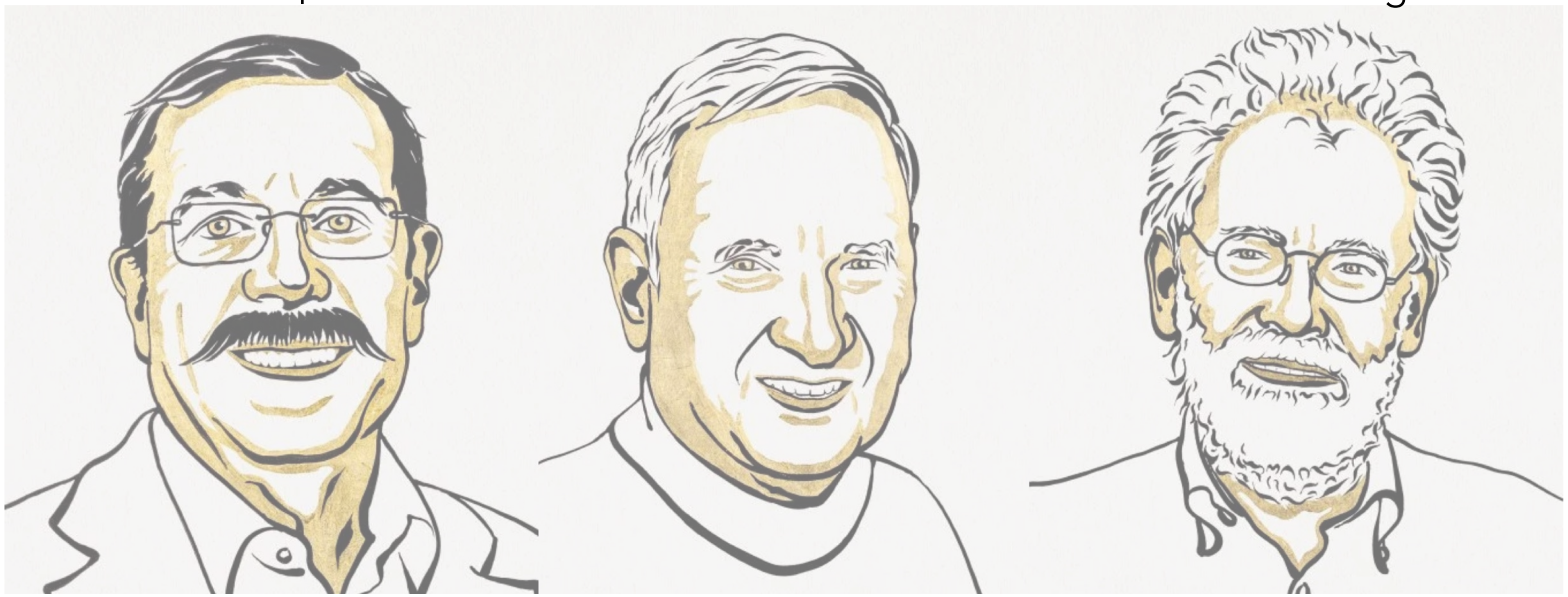
Some remarks on entanglement and nonseparability

Not this entanglement…

V. Sattui (Napa Valley)

Nobel Prize in Physics 2022

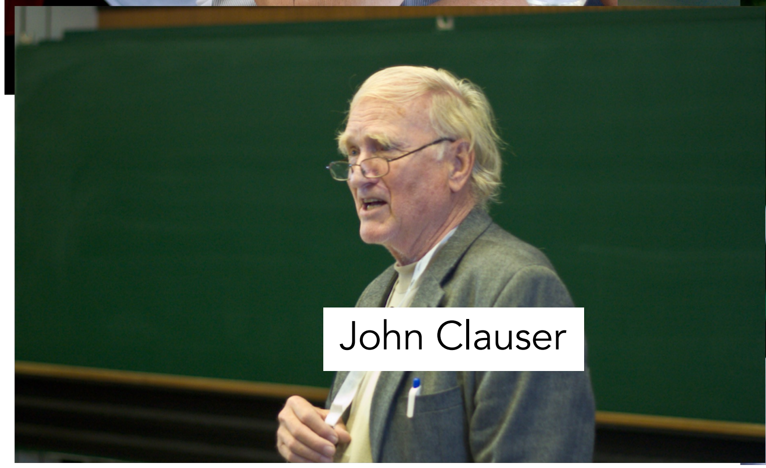
Alan Aspect John Clauser Anton Zeilinger



for experiments with entangled photons,
establishing the violation of Bell inequalities and pioneering quantum information science


Alan Aspect

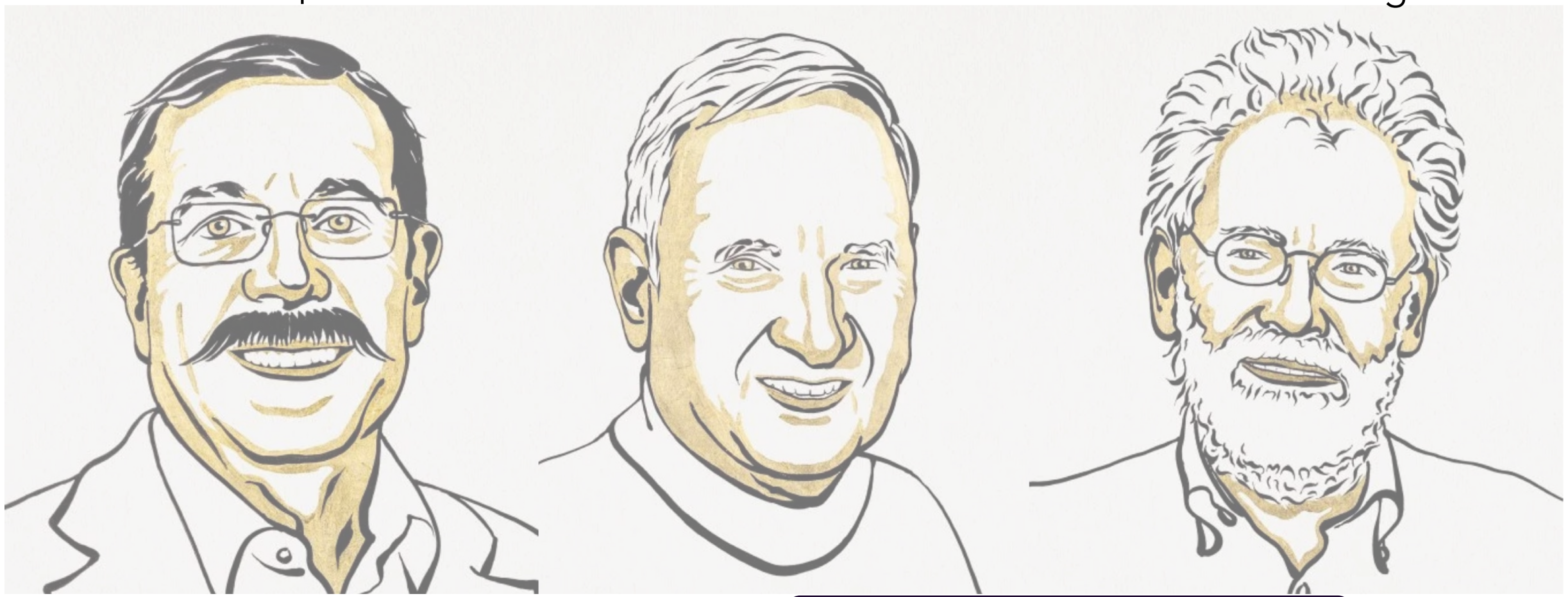

Anton Zeilinger


John Clauser


Mary Bell

Nobel Prize in Physics 2022

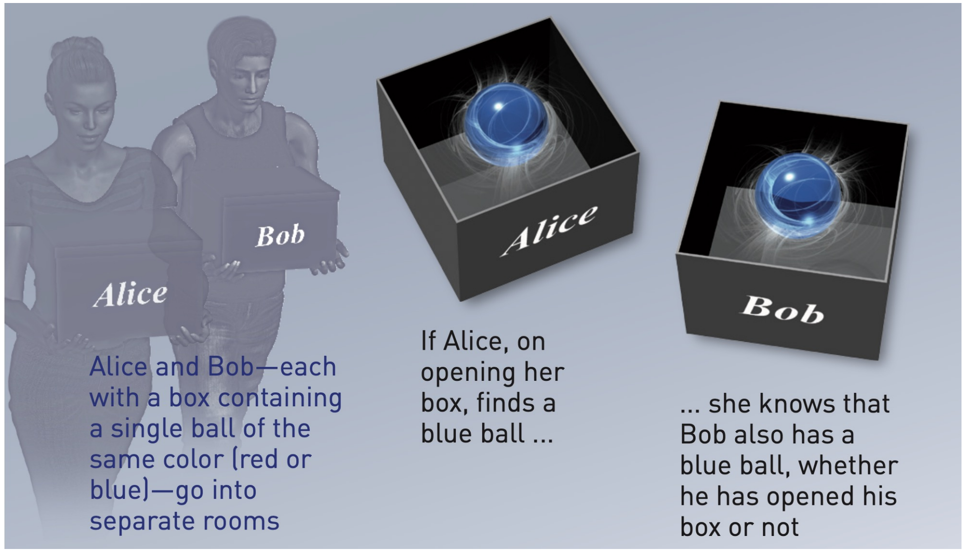
Alan Aspect John Clauser Anton Zeilinger

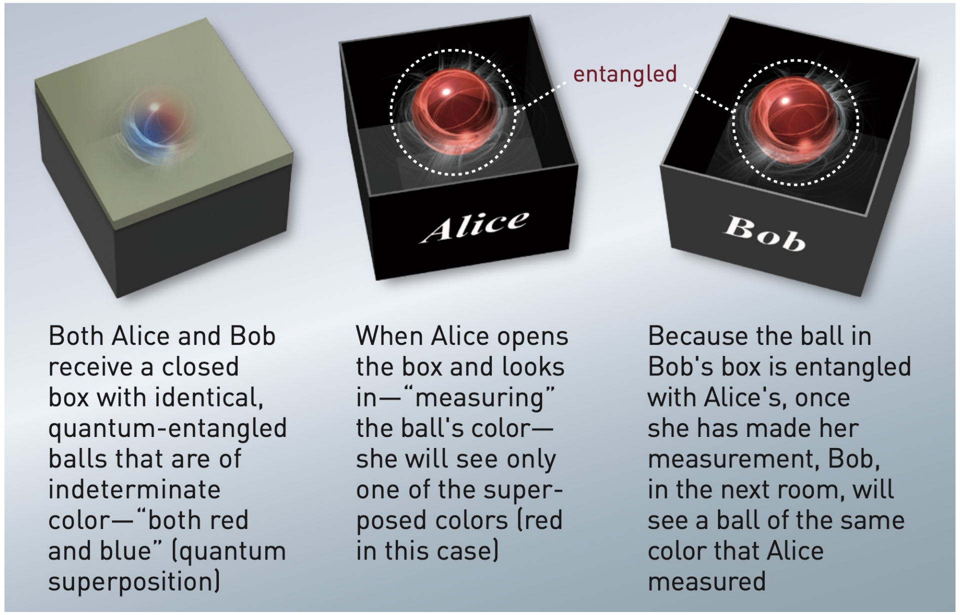


for experiments with entangled photons,
establishing the violation of Bell inequalities and pioneering quantum information science

Entanglement

Correlations that escape classical (common sense) explanation





Repeating this experiment with randomly correlated, ordinary balls, Alice and Bob can make a correlated list.

Repeating this experiment with identical entangled balls, Alice and Bob can make a correlated list…

…but the correlation comes from superposition.

Entanglement

Superposition of superpositions

superposition

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

number of outcomes in each part

superposition of superpositions

$$|\psi_{22}\rangle = \alpha|00\rangle + \beta|11\rangle$$

number of parts

$$\begin{bmatrix} 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Entanglement

The state of one one part can no longer be separated from the other part.

In addition to being correlated in colour, the quantum objects can also be correlated in their shape.

Alice and Bob can choose to measure either the colour or the shape.

If Alice and Bob both choose to measure the same property (either colour or shape), they will always get the correlated results.

If Alice chooses to measure colour, but Bob chooses to measure shape, Bob will see either a ball or a cube, his results are random.

If Alice chooses to measure shape, but Bob chooses to measure colour, Bob will see either red or blue, his results are random.

The state of Bob's object cannot be separated anymore from the state of Alice's object—these objects are entangled, no classical counterpart!

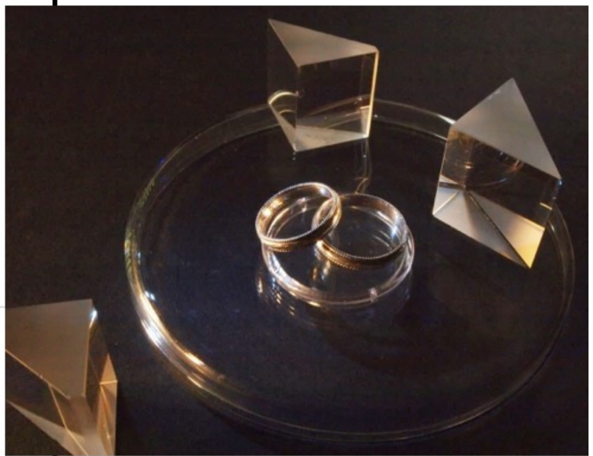
08-04-15 | WORLD CHANGING IDEAS

Instead Of A Wedding, Couples Can Now Opt For Quantum Entanglement

The latest wacky Vegas wedding has couples tying the quantum physics knot–and rethinking what it means to truly be coupled.

BY ADELE PETERS 3 MINUTE READ

A Las Vegas hotel is experimenting with a different kind of marriage ceremony: Instead of hiring a justice of the peace, you can now be married by the power of quantum physics.



Einstein, Podolsky, Rosen 1935 – QM is incomplete

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

Ivan Kaminoff, flickr.com

While we have shown quantum mechanics does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe however, that such a theory is possible.

THE UNIVERSITY OF QUEENSLAND
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Bell 1964: Can local hidden variables explain quantum correlations?

Realism: Measurement results are determined by "hidden variables" that exist prior to and independent of the experimental situation.

Locality: Results obtained at one location are independent of any measurements, or actions at another location.

Freedom of choice: Experimenters are free to choose which measurements to perform.

These assumptions are incompatible with quantum mechanics.

Physics Vol. 1, No. 3, pp. 195–200, 1964 Physics Publishing Co. Printed in the United States

ON THE EINSTEIN PODOLSKY ROSEN PARADOX*

J. S. BELL†
Department of Physics, University of Wisconsin, Madison, Wisconsin

(*Received 4 November 1964*)

NO!

I. Introduction

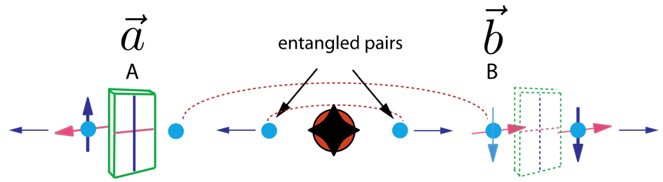
THE paradox of Einstein, Podolsky and Rosen [1] was advanced as an argument that quantum mechanics could not be a complete theory but should be supplemented by additional variables. These additional variables were to restore to the theory causality and locality [2]. In this note that idea will be formulated mathematically and shown to be incompatible with the statistical predictions of quantum mechanics. It is the requirement of locality, or more precisely that the result of a measurement on one system be unaffected by operations on a distant system with which it has interacted in the past, that creates the essential difficulty. There have been attempts [3] to show that even without such a separability or locality requirement no "hidden variable" interpretation of quantum mechanics is possible. These attempts have been examined elsewhere [4] and found wanting. Moreover, a hidden variable interpretation of elementary quantum theory [5] has been explicitly constructed. That particular interpretation has indeed a grossly non-local structure. This is characteristic, according to the result to be proved here, of any such theory which reproduces exactly the quantum mechanical predictions.

Bell's inequality

ON THE EINSTEIN PODOLSKY ROSEN PARADOX*

J. S. BELL†
Department of Physics, University of Wisconsin, Madison, Wisconsin

(*Received* 4 *November* 1964)



Let a complete specification be parametrised by hidden variables: λ

$$\rho(\lambda) \geq 0$$

$$\int d\lambda \rho(\lambda) = 1$$

Measurement outcomes: $A(\vec{a}, \lambda) = \pm 1, B(\vec{b}, \lambda) = \pm 1$

Correlation (product of expectation values): $P(\vec{a}, \vec{b}) = \int d\lambda \rho(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda)$


John Bell

If there is perfect correlation such that: $A(\vec{a}, \lambda) = -B(\vec{a}, \lambda)$

(Original) Bell's inequality: $1 + P(\vec{b}, \vec{c}) \geq |P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c})|$

Correlations due to local hidden variables follow Bell's inequality.

Because correlations predicted by quantum mechanics do not follow Bell's inequality, *no local hidden variable theory would be able to reproduce the predictions of quantum mechanics.*

Bell's inequality

ON THE EINSTEIN PODOLSKY ROSEN PARADOX*

J. S. BELL†
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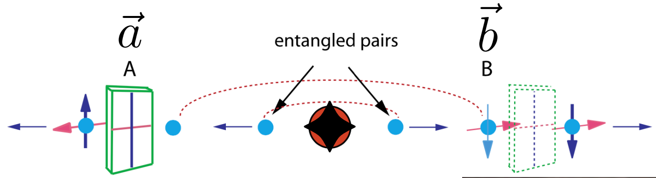

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$$\int d\lambda \rho(\lambda) = 1$$

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Correlation (product of expectation values:

$$P(\vec{a}, \vec{b}) = \int d\lambda \rho(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda)$$

If there is perfect correlation such that: $A(\vec{a}, \lambda) = -B(\vec{a}, \lambda)$ Unrealistic!

(Original) Bell's inequality: $1 + P(\vec{b}, \vec{c}) \geq |P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c})|$

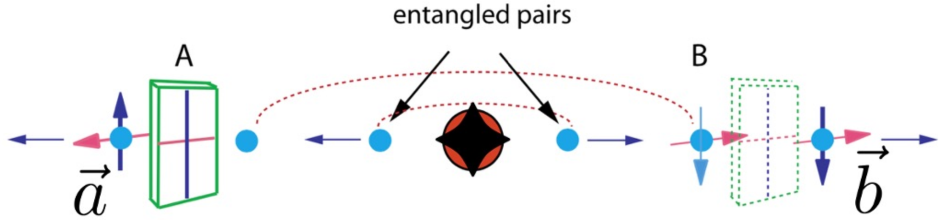
Correlations due to local hidden variables follow Bell's inequality.

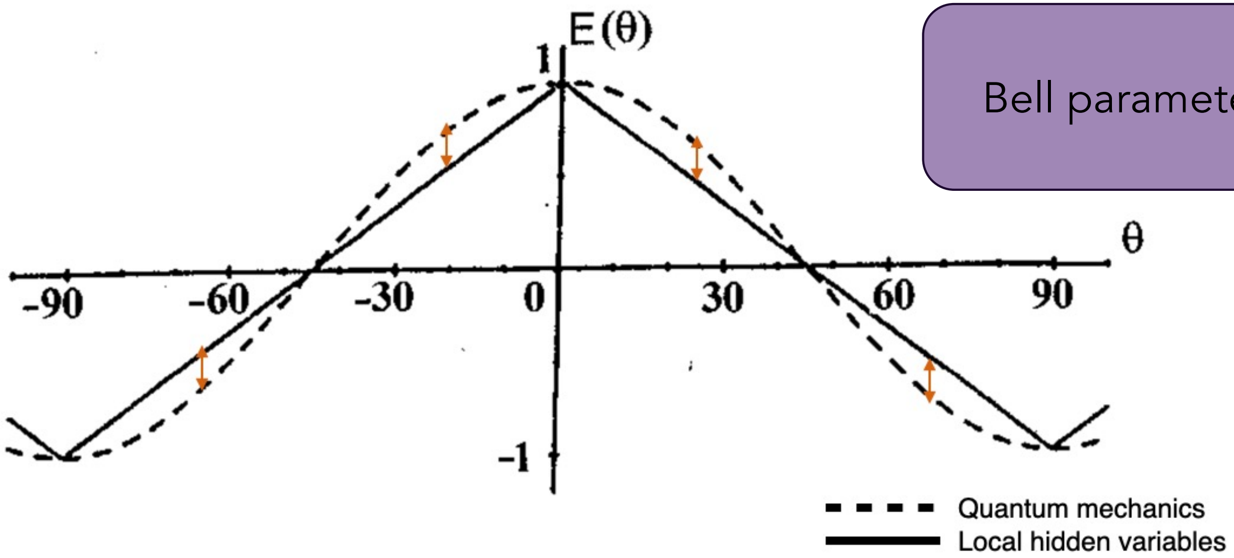
Because correlations predicted by quantum mechanics do not follow Bell's inequality, *no local hidden variable theory would be able to reproduce the predictions of quantum mechanics.*

CHSH version: $|P(a, b) - P(a, c)| \leqslant 2 - P(b', b) - P(b', c)$

Clauser, Horne, Shimony, Holt , PRL 1969

Bell's inequality

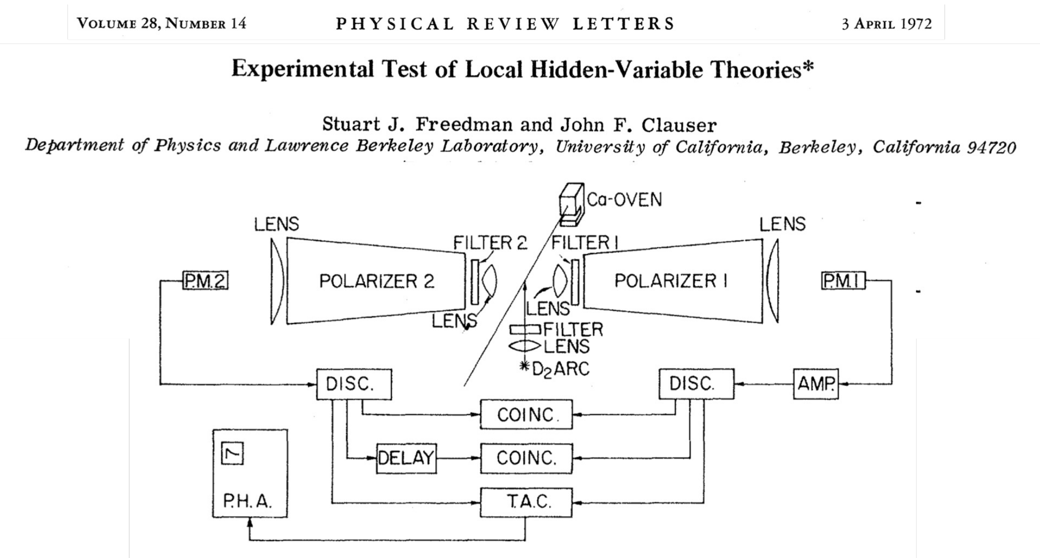


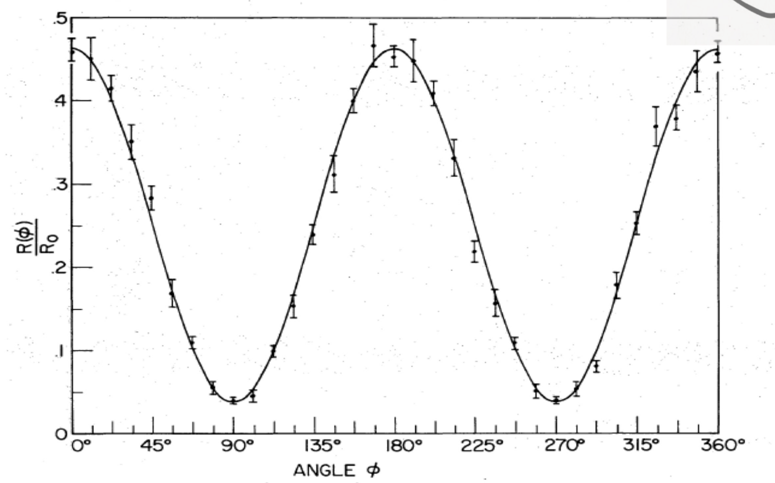


Bell parameter: $|S_{LHV}| \leq 2$

$$|S_{QM}| = 2\sqrt{2}$$

1972: Bell inequality violation in photons entangled in their polarisation





Local hidden variable theories cannot explain quantum correlations.
The correlations measured agree with quantum mechanics.

Gisin's theorem: All entangled pure state of two qubits violate a Bell inequality.

Gisin, Phys. Lett. A, 154 (1991)

Aside: Beyond Tsirelson bound correlations (?)

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Tailored two-photon correlation and fair-sampling: a cautionary tale

J Romero, D Giovannini, D S Tasca, S M Barnett and M J Padgett

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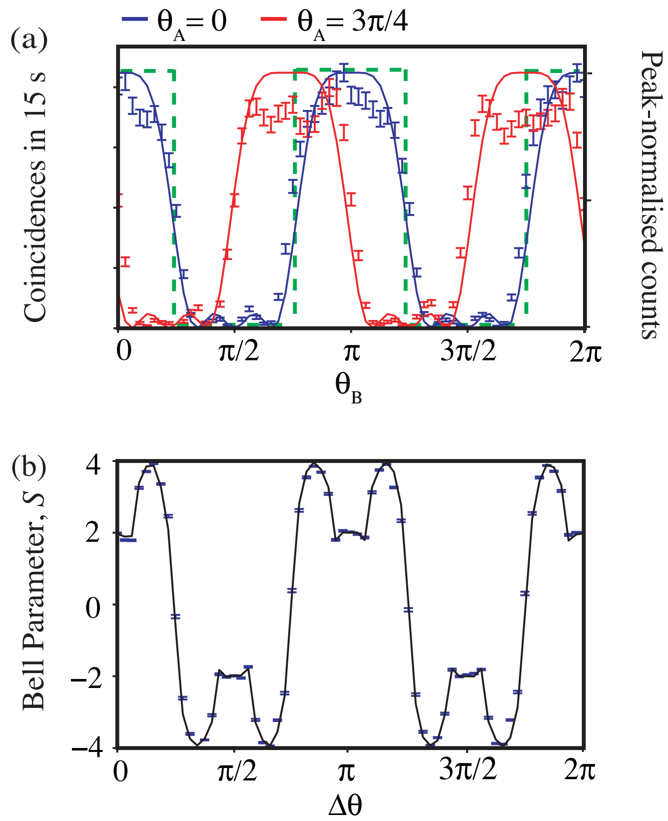
DOI 10.1088/1367-2630/15/8/083047



Aside: "Photon" was a controversial word

Appl. Phys. B 60, 77–84 (1995)

Applied Physics B Lasers and Optics

Anti-photon

W.E. Lamb, Jr.

Optical Sciences Center, University of Arizona, Tucson, AZ 85721, USA

Received: 23 July 1994 / Accepted: 18 September 1994

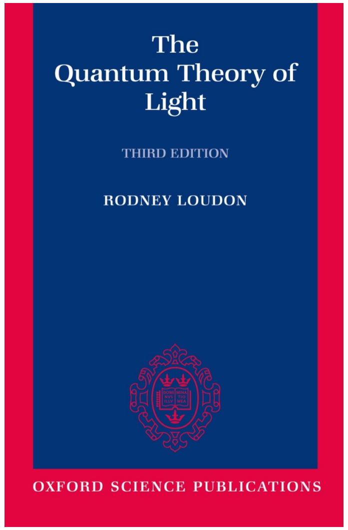
Abstract. It should be apparent from the title of this article that the author does not like the use of the word "photon", which dates from 1926. In his view, there is no such thing as a photon. Only a comedy of errors and historical accidents led to its popularity among physicists and optical scientists. I admit that the word is short and convenient. Its use is also habit forming.

Aside: "Photon" was a controversial word

Introduction: The photon

The use of the word 'photon' to describe the quantum of electromagnetic radiation can lead to confusion and misunderstanding. It is often used in the context of interference experiments, for example Young's slits, in such phrases as 'which slit does the photon pass through?' and 'where do the photons hit the screen when one of the slits is covered up'. The impression is given of a fuzzy globule of light that travels this way or that way through pieces of optical equipment or that light beams consists of streams of the globules, like bullets from a machine gun. Lamb has even argued that there is no such thing as a photon [1] and he has proposed that the word should be used only under licence by properly qualified people! It is, however, difficult to disagree with some of his concerns.

Nevertheless, the word is extremely convenient and its avoidance often leads to lengthy circumlocutions. The adoption of the photon by the quantum-optics community is widespread and the present book follows current usage, with sometimes imprecise statements that could amount to misuse of the word. The intention of this Introduction is to limit the damage that might otherwise occur by briefly explaining the concept of the photon as used in the text. It should be mentioned that the word itself was invented by Lewis [2], with a meaning quite different from that adopted in subsequent work. The relevant history is well reviewed by Lamb [1].



Plan…

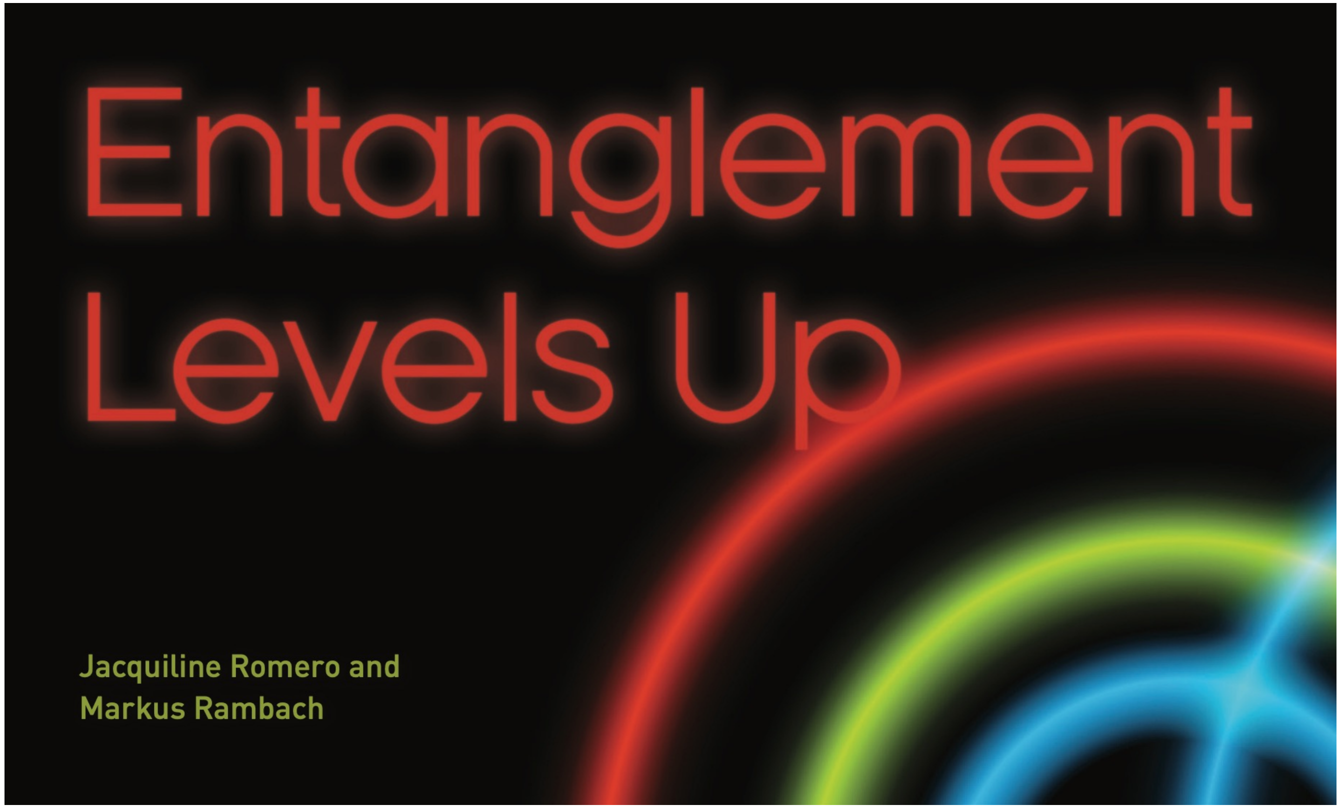
Introduce dits and qudits

Communications tasks: how are dits and qudits similar, and different

Introduce entanglement

Introduce high-dimensional entanglement

Some remarks on entanglement and nonseparability



Optics and Photonics News, Jan 2022

High-dimensional entanglement: many qubits

The 2-qubit states are the Bell states. Local operations on one qubit allows transformation from one Bell state to another.

For systems with more than 2 qubits, it is *not usually* the case that local operations allows transformations from one entangled state to another.

Example: 3-qubit states

Greenberger-Horne-Zeilinger (GHZ) states: $(|000\rangle + |111\rangle)/\sqrt{2}$

Dür-Vidal-Cirac (W) states: $(|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$

Exercise 2 (5-ish minutes)

Which 3-qubit entangled state is resilient to qubit loss, i.e. the state remains entangled even if one qubit is lost.

GHZ state: goodbye elements of reality

For an entangled pair of qubits, Bell's inequality shows how "elements of reality" (hidden variables) are incompatible with quantum mechanics.

GHZ states also demonstrate "elements of reality" are incompatible with quantum mechanics. (…back to the board…)

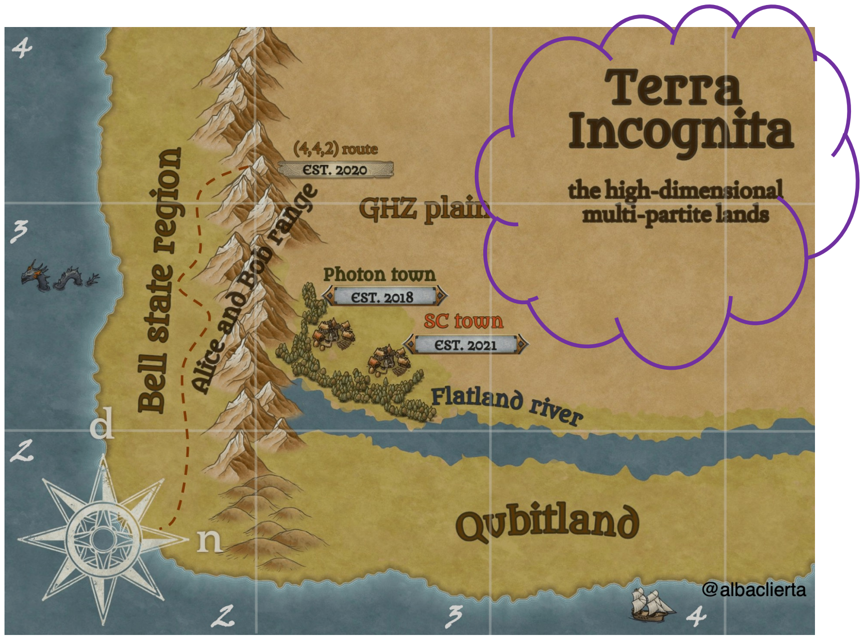
GHZ states demonstration is better in that "elements of reality" can be shown to be incompatible with quantum mechanics in one run of the experiment.

High-dimensional entanglement: many qubits

3-particle entanglement

GHZ state $(|000\rangle + |111\rangle)/\sqrt{2}$

W state $(|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$



4-particle entanglement

$$
\begin{aligned}
G_{abcd} =& \frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{2}(|0011\rangle + |1100\rangle) \\
& + \frac{b+c}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|0110\rangle + |1001\rangle) \\
L_{abc_2} =& \frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle + |1100\rangle) \\
& + c(|0101\rangle + |1010\rangle) + |0110\rangle \\
L_{a_2b_2} =& a(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1010\rangle) \\
& + |0110\rangle + |0011\rangle \\
=& a(|0000\rangle + |1111\rangle) + \frac{a+b}{2}(|0101\rangle + |1010\rangle) \\
L_{ab_3} =& + \frac{a-b}{2}(|0110\rangle + |1001\rangle) \\
& + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle) \\
=& a(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) \\
& + (i|0001\rangle + |0110\rangle - i|1011\rangle) \\
L_{a_4} & \\
=& a(|0000\rangle + |1111\rangle) + (|0011\rangle + |0101\rangle + |0110\rangle) \\
L_{a_2 0_{3\oplus\bar{1}}} =& |0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle \\
L_{0_{5\oplus\bar{3}}} =& \\
L_{0_{7\oplus\bar{1}}} =& |0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle \\
=& |0000\rangle + |0111\rangle
\end{aligned}
$$

Vestraete et al. PRA 2002

High-dimensional entanglement: two or more qudits

Certifying entanglement via quantum state tomography becomes impractical quickly.

Certifying entanglement via entanglement witnesses, and putting a lower bound to dimensionality, are more practical and will be necessary.

Counterpart to the Bell inequality: Collin, Gisin, Linden, Massar, Popescu (CGLMP) inequality.

Collins et a. Phys. Rev. Lett. 040404 (2002)

Entanglement and Schmidt decomposition

The Schmidt decomposition of an entangled state into orthornomal bases $\{u_n\}$ and $\{v_n\}$:

$$|\Psi\rangle = \sum_{n=1}^{r} c_n |u_n\rangle \otimes |v_n\rangle$$

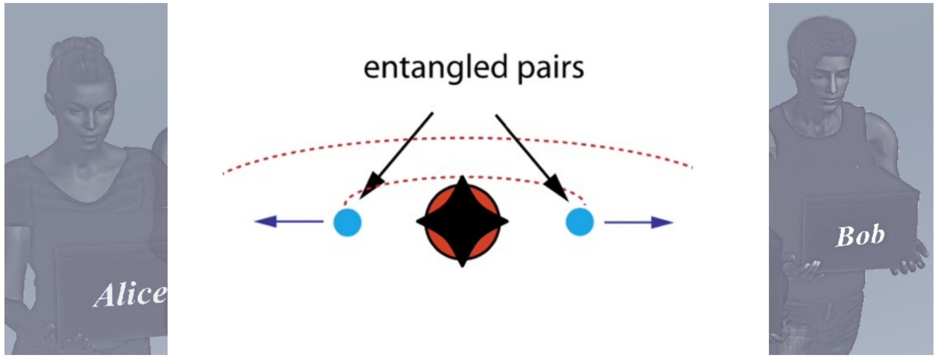
r is the rank of the Schmidt decomposition (number of nonzero coefficients).

Mathematicians/engineers call Schmidt decomposition singular value decomposition (SVD).

The dimensionality of two-qudit entanglement is given by the Schmidt rank (d x d entanglement).

CGLMP Inequality (2 x 2 x d scenario)

Alice chooses to measure A_1 or A_2, each with *d* possible outcomes.



Bob chooses to measure B_1 or B_2, each with *d* possible outcomes.

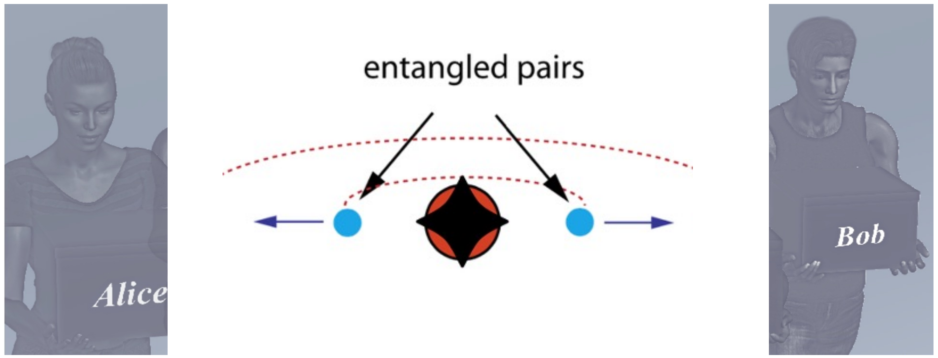
Joint probability of outcomes i for Alice and j for Bob, $i, j \in \{0 \ldots d-1\}$
given Alice chose A_a and Bob chose B_b , $a, b \in \{1,2\}$

quantum prediction: $$\mathbb{P}_Q(i, j \mid a, b) = \mathrm{Tr}\left(A_a^i \otimes B_b^j |\psi\rangle\langle\psi|\right)$$

local hidden variable prediction: $$\mathbb{P}_L(i, j \mid a, b) = \sum_\lambda p(\lambda)\mathbb{P}(i \mid a, \lambda)\mathbb{P}(j \mid b, \lambda)$$

Collins et al. Phys. Rev. Lett. 040404 (2002)
Zoren and Gill, Phys. Rev. Lett. 120406 (2008)

CGLMP Inequality (2 x 2 x d scenario)



$i, j \in \{0 \ldots d-1\}$ $\qquad$ $a, b \in \{1,2\}$

$$\mathbb{P}_Q(i,j \mid a,b) = \mathrm{Tr}\left(A_a^i \otimes B_b^j |\psi\rangle\langle\psi|\right)$$

$$\mathbb{P}_L(i,j \mid a,b) = \sum_{\lambda} p(\lambda)\mathbb{P}(i \mid a,\lambda)\mathbb{P}(j \mid b,\lambda)$$

(simplified) CGLMP inequality:

$$\mathbb{P}_L\left(A_2 < B_2\right) + \mathbb{P}_L\left(B_2 < A_1\right) + \mathbb{P}_L\left(A_1 < B_1\right) + \mathbb{P}_L\left(B_1 \leq A_2\right) \geqslant 1$$

where, $\qquad \mathbb{P}_L\left(A_a < B_b\right) = \sum_{i<j} \mathbb{P}_L(i,j \mid a,b)$

Collins et al. Phys. Rev. Lett. 040404 (2002)
Zoren and Gill, Phys. Rev. Lett. 120406 (2008)

CGLMP Inequality (2 x 2 x d scenario)

$$\mathbb{P}_L\left(A_a < B_b\right) = \sum_{i<j} \mathbb{P}_L(i, j \mid a, b)$$

$$\mathbb{P}_L\left(A_2 < B_2\right) + \mathbb{P}_L\left(B_2 < A_1\right) + \mathbb{P}_L\left(A_1 < B_1\right) + \mathbb{P}_L\left(B_1 \leq A_2\right) \geqslant 1$$

(conjecture) measurements to maximally violate the CGLMP inequality with maximally entangled states:

$$|i\rangle_{A,a} = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \exp\left(\mathbf{i}\frac{2\pi}{d} k\left(i + \alpha_a\right)\right) |k\rangle_A$$

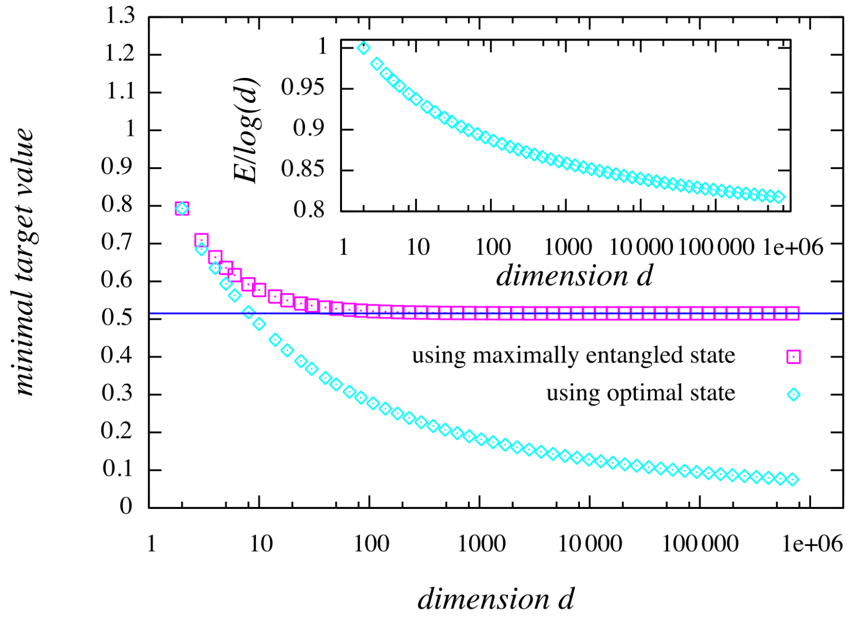
$$|j\rangle_{B,b} = \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} \exp\left(\mathbf{i}\frac{2\pi}{d} l\left(-j + \beta_b\right)\right) |l\rangle_B$$

$$\alpha_1{=}0, \alpha_2{=}1/2, \beta_1{=}1/4, \beta_2{=}-1/4$$

Numerically, one can optimise over measurements and and entangled states.

Collins et al. Phys. Rev. Lett. 040404 (2002)
Zoren and Gill, Phys. Rev. Lett. 120406 (2008)

CGLMP Inequality (2 x 2 x d scenario)



The conjecture on what the optimal measurements seems to hold.

The optimal state for maximum violation is not the maximally entangled state.

TABLE I. Violation of the CGLMP inequality.

d	$\min \mathcal{A}$	λ_0	λ_1	λ_2	λ_3	λ_4
2	0.7929	0.7071	0.7071	⋯	⋯	⋯
3	0.6950	0.6169	0.4888	0.6169	⋯	⋯
4	0.6352	0.5686	0.4204	0.4204	0.5686	⋯
5	0.5937	0.5368	0.3859	0.3859	0.3859	0.5368

Collins et al. Phys. Rev. Lett. 040404 (2002)
Zoren and Gill, Phys. Rev. Lett. 120406 (2008)

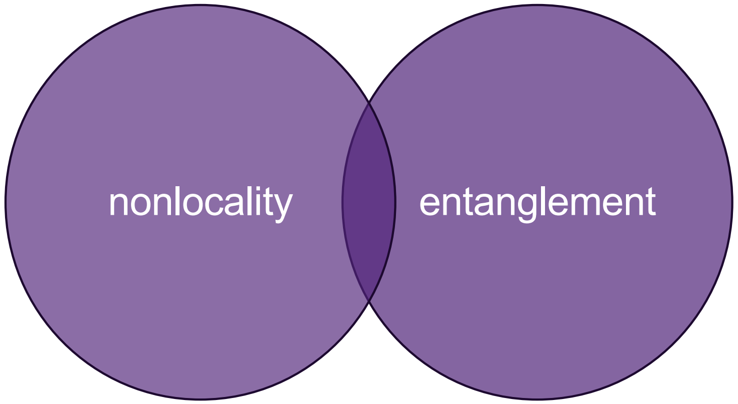
CGLMP Inequality (2 x 2 x d scenario)

Maximal violations of the (CHSH)-Bell inequality are demonstrated by using maximally entangled states of qubits.

Maximal violations of the CGLMP inequality are demonstrated by using non-maximally entangled states of qubits.

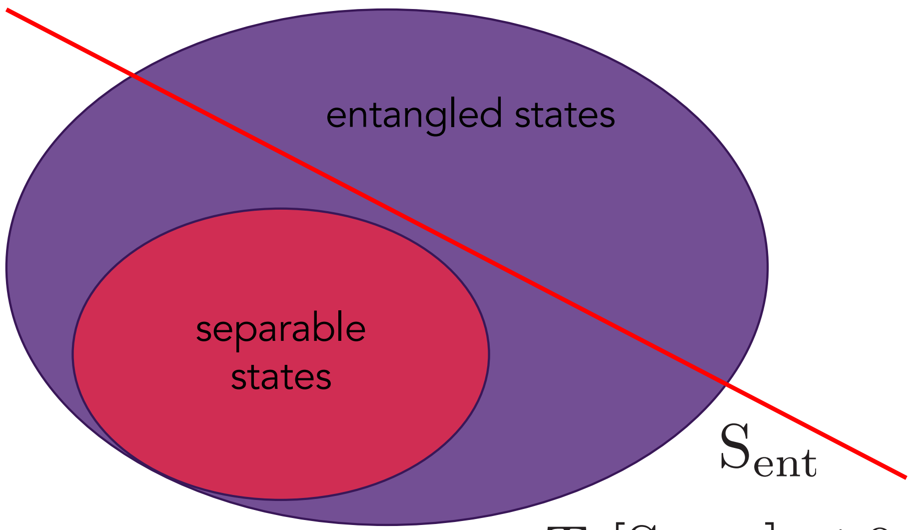
With entangled qudits, nonlocality and entanglement as signatures of nonclassicality are clearly separated.

CGLMP inequality can be used as a device-independent dimension witness.



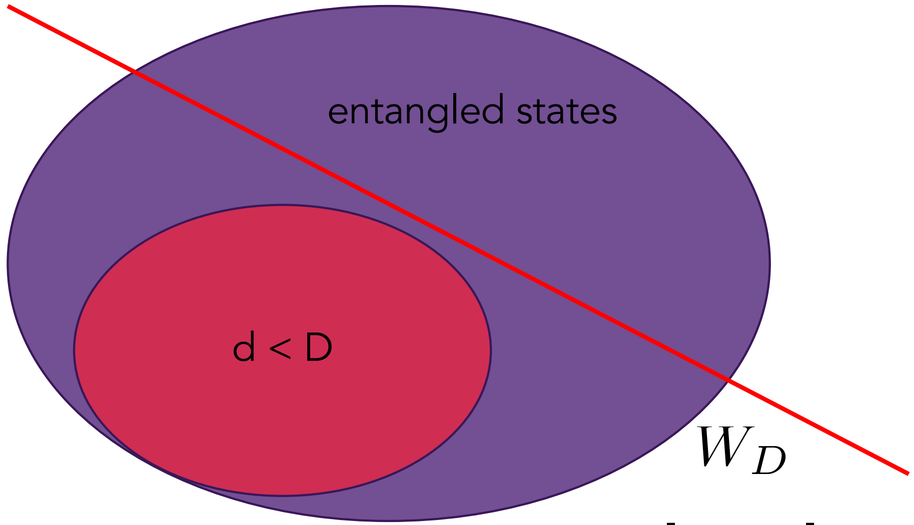
Collins et al., Phys. Rev. Lett. 040404 (2002)
Zoren and Gill, Phys. Rev. Lett. 120406 (2008)
Cai et al., J. Phys. A, 305301 (2016)

Entanglement and dimension witnesses (2 x 2 x d scenario)



$$\mathrm{Tr}[\mathrm{S}_{\mathrm{ent}}\rho] \leq 0$$

for entangled states



$$\mathrm{Tr}[\mathrm{W}_{\mathrm{D}}\rho] \leq \mathrm{I}_{\mathrm{D}}$$

for d x d entangled states

Cai et al., J. Phys. A, 305301 (2016)

Entanglement and dimension witnesses (2 x 2 x d scenario)

If there is some knowledge of the state, e.g. that the state is pure, or close to pure state, $|\Phi\rangle$ or a dephasing of $|\Phi\rangle$, the fidelity of state ρ can be bounded from below:

$$|\Phi\rangle = \sum_{m=0}^{d-1} \lambda_m |mm\rangle$$

$$\tilde{F}(\rho, \Phi) \leq \boxed{F(\rho, \Phi) \leq B_k(\Phi)}$$

$\{|ij\rangle\}$ $\{|mn\rangle\}$

"tilted"

$$B_k(\Phi) := \sum_{m=0}^{k-1} \lambda_{i_m}^2$$

Inequality is satisfied by states with Schmidt rank k or less, if the fidelity is greater, the system has local dimension of at least *k+1*.

Bavaresco et al., Nature Phys. 14, 1032 (2018)

Plan…

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Some remarks on entanglement and nonseparability

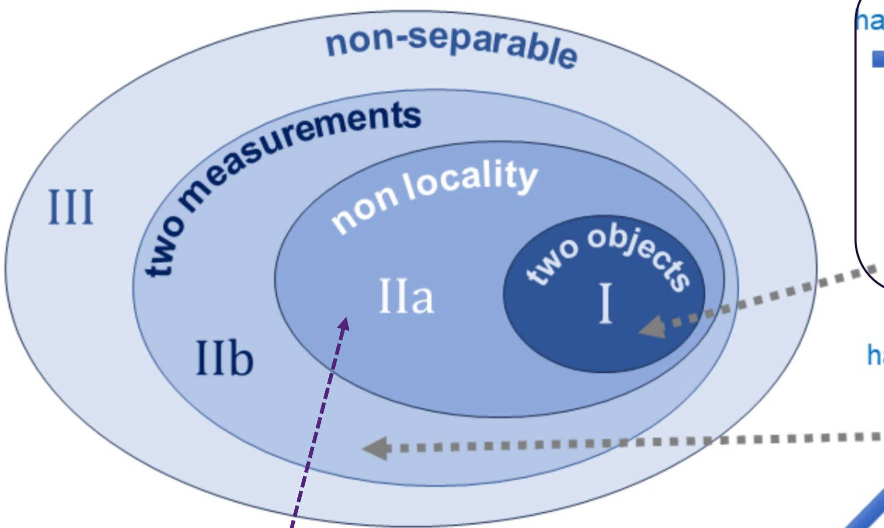
Korolkova et al., Philosophical Trans. A. 382, 28227 (2024)

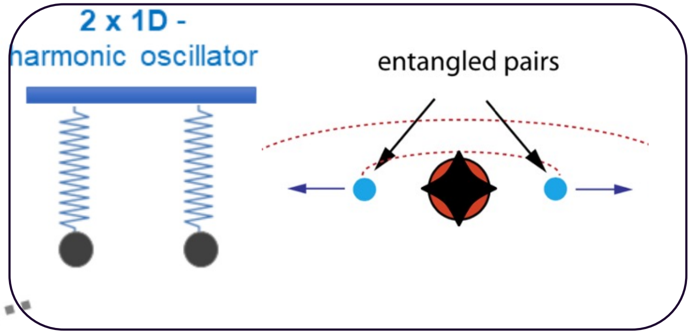
Entanglement and nonseparability: quantum and classical

Quantum entanglement refers only to statistical correlations between measurement outcomes on two partitions of the Hilbert space.

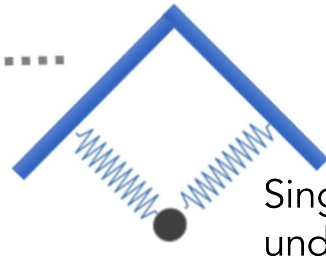
physical

mathematical



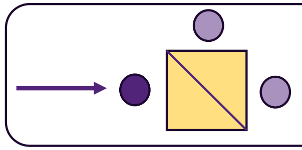


1 x 2D -
harmonic oscillator



$$|\Psi_{2\times 1\mathrm{D}}\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|n\rangle_A|m\rangle_B - |m\rangle_A|n\rangle_B\right)$$

$$|\Psi_{1\times 2\mathrm{D}}\rangle_{xy} = \frac{1}{\sqrt{2}}\left(|n\rangle_x|m\rangle_y - |m\rangle_x|n\rangle_y\right).$$



Single excitation shared by 2 mode functions

Single particle undergoing 2 excitations

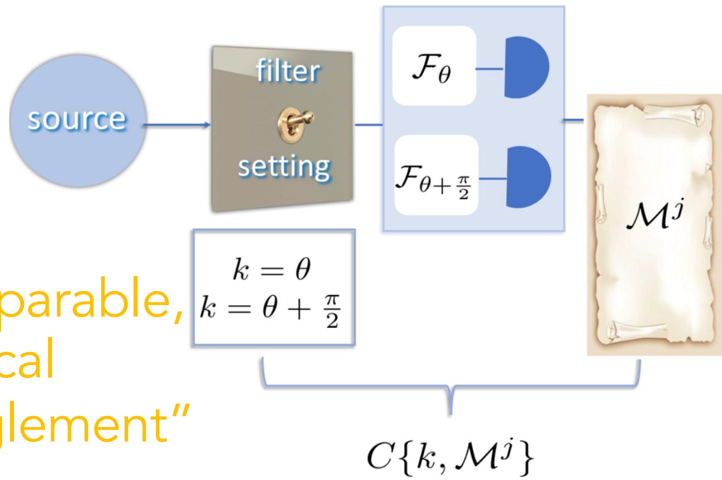
Nonseparability

$$|\Psi\rangle = |\text{ excitation }\rangle_{\text{degrees of freedom}}$$

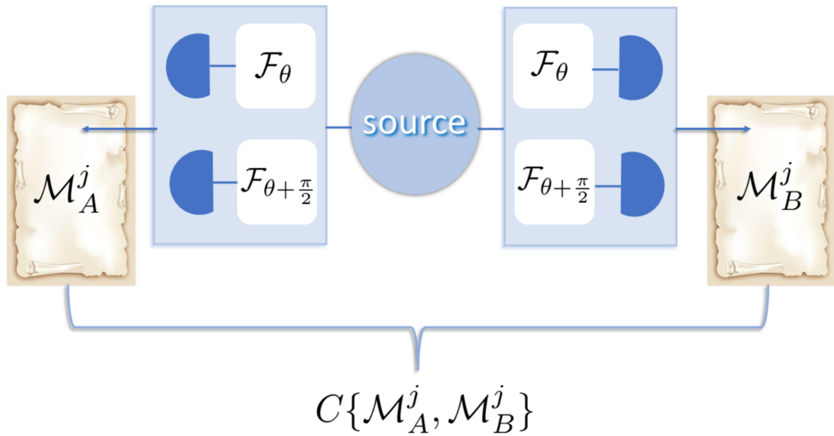
Mode functions in optics are "frames of excitation"

$$|\Psi\rangle \propto [|\psi_k\rangle_1 |\psi_l\rangle_2 - |\psi_{k'}\rangle_1 |\psi_{l'}\rangle_2] |E(\mathbf{r}, t)\rangle$$

1,2, refer to DOFs; k,k' and l,l' refer to mode functions



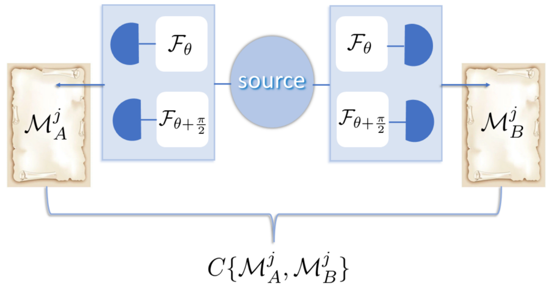
Nonseparable, "classical entanglement"



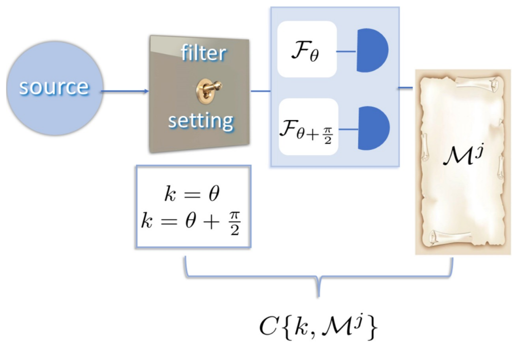
Korolkova et al., Philosophical Trans. A. 382, 28227 (2024)

Entanglement is about "decisive" nonseparability

Entanglement requires nonseparability that leads to statistically correlated quantum measurements.



$$|\Psi\rangle \propto |\psi_{LG1}(\mathbf{r})\rangle_A |H\rangle_B - |\psi_{LG2}(\mathbf{r})\rangle_A |V\rangle_B$$



There is a deterministic correlation between the filtering (which is unitary) and the measurement outcome.

There is a single measurement outcome based on some choice of k.

Korolkova et al., Philosophical Trans. A. 382, 28227 (2024)

The University of Queensland Australia

Take home…

- Qudits are described by complex numbers in vectors or matrices. Using qudits is a double-edged sword.
- Measuring a qudit results to one dit, same as measuring a dit, but, there are communication tasks where qudits can give advantage over using dits.
- Entanglement is a superposition of superpositions.
- High-dimensional entanglement offers a bigger playground for quantum foundations and quantum technologies. CGLMP violations are more robust.
- Entanglement is not just nonseparability, two measurements are required. (This is a subtlety that is most relevant to photons, which has many accessible modes.)