Schurian-infinite blocks of Hecke algebras of types A and B

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Hecke algebras

The type A Hecke algebra is the unital, associative $\mathbb{F}$-algebra $\mathscr{H}_n$ with generators $T_1, T_2, \ldots, T_{n-1}$ and relations

$$\begin{aligned}(T_i - q)(T_i + 1) &= 0 && \text{for all } i,\\ T_i T_j &= T_j T_i && \text{for } |i-j| > 1,\\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \leqslant i \leqslant n-2,\end{aligned}$$

where $q \in \mathbb{F}$ is a primitive eth root of unity.
$\mathscr{H}_n$ is semisimple if $e > n$.

The *Specht modules* $\{\mathbf{S}^\lambda \mid \lambda \vdash n\}$ over $\mathscr{H}_n$ are the ordinary irreducible $\mathscr{H}_n$-modules, indexed by partitions λ of n.

If $e \leqslant n$, the simple modules appear as quotients of the Specht modules: $\{\mathrm{D}^\lambda \mid \lambda \vdash n,\ \lambda \text{ is } e\text{-regular}\}$.

Hecke algebras

The type B Hecke algebra $\mathscr{H}_{Q,n}$ is a deformation of $(\mathbb{Z}/2\mathbb{Z}) \wr \mathfrak{S}_n$.
It has gens $T_0, T_1, T_2, \dots, T_{n-1}$ and rels

$$
\begin{aligned}
(T_0 - Q_1)(T_0 - Q_2) &= 0 & T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\
(T_i - q)(T_i + 1) &= 0 & &\text{for all } i \neq 0, \\
T_i T_j &= T_j T_i & &\text{for } |i - j| > 1, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} & &\text{for } 1 \leqslant i \leqslant n-2,
\end{aligned}
$$

where $q \in \mathbb{F}$ is a primitive eth root of unity and $Q_i \in \mathbb{F}$.

Hecke algebras

If $Q_i = q^{\kappa_i}$ for some $\kappa_i \in \mathbb{Z}$, then $\mathscr{H}_{Q,n}$ is semisimple if $e > n$ and κ_i are 'far enough apart'. Otherwise, $\mathscr{H}_{Q,n}$ is Morita equivalent to a direct sum of tensor products of type A Hecke algebras.

The *Specht modules* $\{\mathbf{S}^\lambda \mid \lambda \text{ a bipartition of } n\}$ over $\mathscr{H}_{Q,n}$ are the ordinary irreducible $\mathscr{H}_{Q,n}$-modules.

When $\mathscr{H}_{Q,n}$ is not semisimple, the simple modules appear as quotients of the Specht modules: $\{\mathrm{D}^\lambda \mid \lambda \vdash_2 n,\ \lambda \text{ is } \textbf{Kleshchev}\}$.

Schurian-finiteness

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \geqslant 0$ throughout. For any $\mathbb{F}$-algebra A, we say that an A-module M is **Schurian** (or a brick) if $\mathrm{End}_A(M) \cong \mathbb{F}$. We say that A is **Schurian-finite** (brick-finite) if there are only finitely many isomorphism classes of Schurian A-modules, and **Schurian-infinite** (brick-infinite) otherwise.

Schurian modules must be indecomposable, so clearly

$$\text{representation-finite} \Rightarrow \text{Schurian-finite}.$$

The converse is not true in general – e.g. preprojective algebras of type other than A_n for $1 \leqslant n \leqslant 4$ are representation-infinite, but Schurian-finite.

Schurian-finiteness

A result of Demonet, Iyama and Jasso (2019) yields that A is Schurian-finite if and only if it is τ-tilting finite.

So we can use established results for τ-tilting (in)finite algebras to determine when algebras are Schurian-(in)finite. In particular, we make heavy use of the following reduction result.

Proposition

If the Gabriel quiver of a finite-dimensional $\mathbb{F}$-algebra A contains the quiver of an affine Dynkin diagram with zigzag orientation (i.e. every vertex is a sink or a source) as a subquiver, then A is Schurian-infinite.

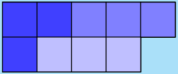
We want to determine the Schurian-finiteness of blocks of Hecke algebras (of types A or B), using the above proposition.

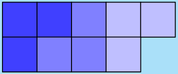
Blocks – type A

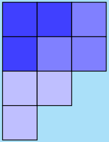
Specht modules S^λ and S^μ (or simple modules D^λ and D^μ) are in the same block of $\mathscr{H}_n$ if and only if λ and μ *have the same core*.

Example

Let $\lambda = (5, 4)$, $\mu = (3^2, 2, 1)$, and $e = 3$. Then λ and μ are in the same block:

$[\lambda] =$ 



$[\mu] =$ 

The *weight* or *defect* of a partition is the number of e-rim hooks that can be removed before obtaining the core. e.g. $w = 3$ above, with core the empty partition.

Blocks – type A

An equivalent description can be given in terms of partitions having *equal multisets of residues modulo e*.

Example

Let $\lambda = (5,4)$, $\mu = (3^2, 2, 1)$, and $e = 3$. Then λ and μ are in the same block:

$$[\lambda] = \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 2 & 0 & 1 \\ \hline 2 & 0 & 1 & 2 \\ \cline{1-4} \end{array} \qquad [\mu] = \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 \\ \cline{1-2} 0 \\ \cline{1-1} \end{array}$$

This description readily generalises to describe the blocks of type B Hecke algebras (for $Q_i = q^{\kappa_i}$).

Blocks – type B

$Q_i = q^{\kappa_i} \rightsquigarrow$ fill bipartitions with residues mod e, starting with κ_i in the top-left corner of component i.
Spechts S^λ and S^μ (or simples D^λ and D^μ) are in the same block of $\mathscr{H}_{Q,n}$ iff bipartitions λ & μ have same multisets of residues mod e.

Example

Let $e = 3$, $\kappa = (0, 1)$, $\lambda = ((3, 1^2), (3, 2^2, 1))$, and $\mu = ((5, 2^2), (2^2))$. Then λ and μ are in the same block:

$[\lambda] =$

0	1	2
2		
1		

1	2	0
0	1	
2	0	
1		

$[\mu] =$

0	1	2	0	1
2	0			
1	2			

1	2
0	1

There is a block invariant called *defect*, but it's more complicated than in type A. Each e-hook contributes 2 to the defect.

Graded decomposition numbers

Results of Brundan, Kleshchev, and Wang $\rightsquigarrow$ $\mathscr{H}_n$ and $\mathscr{H}_{Q,n}$ are isomorphic to cyclotomic KLR algebras, and their Specht modules and simple modules may be graded.

The graded decomposition number $d^p_{\lambda\mu}(v)$ is defined to be the graded composition multiplicity of D^μ in S^λ. In other words

$$d^p_{\lambda\mu}(v) = [S^\lambda : D^\mu]_v = \sum_{d\in\mathbb{Z}} [S^\lambda : D^\mu\langle d\rangle] v^d \in \mathbb{N}[v, v^{-1}].$$

Extensions – type A

Using a result of Shan on Jantzen filtrations and radical filtrations of Weyl modules for q-Schur algebras, we can deduce the following.

Lemma

Suppose that $e \geqslant 3$, $p = 0$, and λ, μ are e-regular partitions of n. If the **coefficient of v** in $d^0_{\lambda\mu}(v)$ is nonzero, then

$$\mathrm{Ext}^1_{\mathscr{H}_n}(\mathrm{D}^\lambda, \mathrm{D}^\mu) = \mathrm{Ext}^1_{\mathscr{H}_n}(\mathrm{D}^\mu, \mathrm{D}^\lambda) \neq 0.$$

Combining with an idempotent truncation argument, we get our main tool for showing that a block of $\mathscr{H}_n$ is Schurian-infinite.

Extensions – type B

Results of Maksimau play a similar role in type B, working with graded lifts of cyclotomic q-Schur algebras, and their Kozul grading.

Lemma

Suppose that $e \geqslant 2$, $p = 0$, and λ, μ are **Kleshchev bipartitions** of n. If the **coefficient of v** in $d^0_{\lambda\mu}(v)$ is nonzero, then

$$\mathrm{Ext}^1_{\mathscr{H}_{Q,n}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) = \mathrm{Ext}^1_{\mathscr{H}_{Q,n}}(\mathrm{D}^\mu, \mathrm{D}^\lambda) \neq 0.$$

Combining with an idempotent truncation argument, we get our main tool for showing that a block of $\mathscr{H}_{Q,n}$ is Schurian-infinite.

Key Proposition (Ariki–Lyle–S., 2023)

Suppose $e \geqslant 2$ & $p \geqslant 0$. If the char 0 graded decomposition matrix has one of the following as a submatrix, and $d^p_{\lambda\mu}(1) = d^0_{\lambda\mu}(1) \in \{0,1\}$ for all row labels λ, μ of the submatrix, then the block is Schurian-infinite.

$$\begin{pmatrix} 1 & & & \\ v & 1 & & \\ 0 & v & 1 & \\ v & v^2 & v & 1 \end{pmatrix} \qquad (\dagger) \qquad\qquad \begin{pmatrix} 1 & & & \\ v & 1 & & \\ v & 0 & 1 & \\ v^2 & v & v & 1 \end{pmatrix} \qquad (\ddagger)$$

Take the matrix $(\ddagger)$, with rows and columns labelled by four e-regular partitions $\lambda, \mu, \nu, \omega$. Then if $p = 0$, the previous lemma gives subquiver

$$\begin{array}{ccc} \lambda & \text{———} & \mu \\ | & & | \\ \nu & \text{———} & \omega \end{array}$$

which is $A_3^{(1)} \rightsquigarrow$ the result (in characteristic 0).

Main result – type A

(It is known that a block of $\mathscr{H}_n$ of weight 0 or 1 is representation-finite and therefore Schurian-finite.)

Theorem (Ariki–Lyle–S., 2023)

Suppose $e \geqslant 3$, and that B is any block of $\mathscr{H}_n$ with weight $\geqslant 2$. Then B is Schurian-infinite in any characteristic.

Hidden in this theorem is A LOT of work. Ingredients include James–Mathas's runner removal, LLT algorithm, a graded analogue of Scopes equivalences, work on (graded) decomposition numbers and Ext^1 by Richards, Fayers, Fayers–Tan, analysis of Specht homomorphisms, ...

Strategy of proof

1) Look at blocks B of fixed weight 2 or 3, and find a submatrix like in the Key Proposition. Find four e-regular partitions in B that will index the submatrix. Display them on an abacus.

2) In characteristic 0, we use 'runner-removal' results to reduce e down to 3 or 4.

3) Compute the corresponding submatrix of the char. 0 graded decomposition matrix for a representative of each 'Scopes class', e.g. by the LLT algorithm.

4) Compute these same graded decomposition numbers in char. p using the adjustment matrix results of Richards and Fayers in weight 2, and of Fayers–Tan in weight 3.

5) For blocks of weight $\geqslant 4$, use a row-removal result to equate graded decomposition numbers to those in weights 2 or 3.

Representation type

Definition

The representation type of an $\mathbb{F}$-algebra A is said to be:

- *finite* if it has finitely many indecomp. modules, up to isom.;
- *tame* if for any d, all but fin. many d-dimensional indecomp. modules lie in fin. many one-parameter families, up to isom.;
- *wild* if $\exists$ a fin.-gen. A–$\mathbb{F}\langle X, Y\rangle$-bimodule M, which is free as a right $\mathbb{F}\langle X, Y\rangle$-module, s.t. the functor $M \otimes_{\mathbb{F}\langle X,Y\rangle} - : \mathbb{F}\langle X, Y\rangle\text{-mod} \to A\text{-mod}$ preserves indecomposability and isomorphism classes.

Theorem (Drozd, 1979)

Any $\mathbb{F}$-algebra A has representation type that is exactly one of the above three types.

Strictly wild algebras

Definition

An $\mathbb{F}$-algebra A is said to be:

- *wild* if $\exists$ a fin.-gen. A–$\mathbb{F}\langle X, Y\rangle$-bimodule M, which is free as a right $\mathbb{F}\langle X, Y\rangle$-module, s.t. the functor $M \otimes_{\mathbb{F}\langle X,Y\rangle} - : \mathbb{F}\langle X, Y\rangle\text{-mod} \to A\text{-mod}$ preserves indecomposability and isomorphism classes.
- *strictly wild* if the functor above is *full*.

Not every wild algebra is strictly wild. e.g. $\mathbb{F}[x, y, z]/(x, y, z)^2$ is wild, but not strictly wild.

Fact

A strictly-wild algebra is Schurian-infinite (brick-infinite). In fact, stronger still, a strictly-wild algebra is actually **brick-wild**.

Refined classification

Theorem (S., 2024)

Suppose $e \geqslant 3$, and that B is any block of $\mathscr{H}_n$ with weight $\geqslant 2$. If $e = 3$, suppose further that B is not (Scopes equivalent to) the weight 2 Rouquier block. Then B is strictly wild, and therefore brick-wild, in any characteristic.

I don't know if the result holds for that one remaining block! Our methods don't work, at least.

Main result – type B

(It is known that a block of $\mathscr{H}_{Q,n}$ of weight 0 or 1 is representation-finite and therefore Schurian-finite.)

First, suppose that $Q_i \neq q^{\kappa_i}$.

Then $\mathscr{H}_{Q,n}$ is Morita equivalent to a direct sum of tensor products of algebras $\mathscr{H}_m$. Under this equivalence, any block of $\mathscr{H}_{Q,n}$ goes to a tensor product of 'type A blocks' B and B'.

Proposition (Ariki–Lyle–S.–Wang, 2025)

If $e \geqslant 3$, then $B \otimes B'$ is Schurian-finite if and only if $\mathrm{weight}(B) + \mathrm{weight}(B') \leqslant 1$ (if and only if that block has finite representation type).

Main result – type B

From now on, we assume that $Q_i = q^{\kappa_i}$.

Theorem-in-progress (Ariki–Lyle–S.–Wang, 2025)

Suppose $e \geqslant 4$, and that B is any block of $\mathscr{H}_{Q,n}$ with defect $\geqslant 2$. Then B is Schurian-infinite in any characteristic.

Suppose $e = 3$, and that B is any block of $\mathscr{H}_{Q,n}$ with defect $\geqslant 3$. Then B is Schurian-infinite in any characteristic.

If $e = 3$, blocks of defect 2 are not so clear-cut, but we can classify which are Schurian-finite. They are all Schurian-finite if $\kappa_1 = \kappa_2$, but otherwise some are Schurian-finite, while others are not (they fall into 6 Scopes classes).

Strategy of proof

'Higher-level runner removal' (Dell'Arciprete(–Putignano) can't get us the decomp. #s we need. **Assume $e \geqslant 4$.**

1) If $\kappa_1 \neq \kappa_2$, can uniformly treat all core blocks of defect $\geqslant 2$ (char-free). Can add e-hooks to those and use 'row-removal' (Bowman–S.). Ditto for $\kappa_1 = \kappa_2$, core blocks of defect $\geqslant 3$.

2) Any block with $\geqslant 2$ removable e-hooks on some bipartitions $\rightsquigarrow$ row-removal again – cut to level 1 and use our type A work.

This leaves core blocks of defect $\leqslant 2$ with 1 hook added on. These have defect 2, 3, or 4.

3) Look at blocks B of defect 2, and find a submatrix like in the Key Proposition. Fayers has given formulae for these decomposition matrices, and they are characteristic-free. Direct computation shows that defect 2 *core blocks* are Schurian-finite when $\kappa_1 = \kappa_2$. Otherwise, they are all Schurian-infinite.

Strategy of proof

4) Decomposition numbers in defect 3 blocks are known to be characteristic-free (and always 0 or 1) by Fayers–Putignano. We have conjectural descriptions of the submatrices we need for $\kappa_1 \neq \kappa_2$.

The $\kappa_1 = \kappa_2$ case remains to be solved.

5) The remaining defect 4 blocks have $\kappa_1 = \kappa_2$, and are a defect 2 core block with a single hook added on. These are difficult – the decomposition numbers aren't characteristic-free, etc.

6) All the $e = 3$ cases need slightly different handling, especially in characteristic 2.