Entropy, cocycles, algebraic K-theory and diagrammatics

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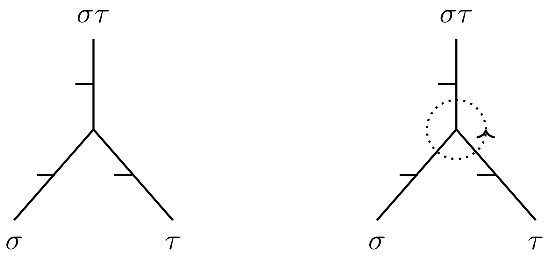
Entropy, cocycles and diagrammatics.

Joint with Mikhail Khovanov.

Let G be a group. Consider lines in the plane with a coorientation, which are labelled by elements in the group.

Lines labelled by σ and τ merge toward the line labelled by $\sigma\tau$.

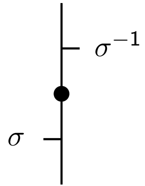
Have monodromy at each vertex.



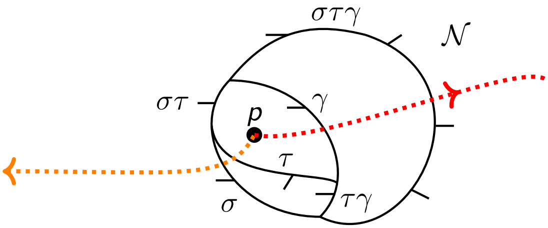
Coorientation moves in the same direction, then the product of these elements, taken with inverses for the opposite orientation, should be $\sigma, \tau, (\sigma\tau)^{-1}$, i.e., $\sigma\tau(\sigma\tau)^{-1} = 1$.

So around this vertex, we have trivial monodromy.

This dot (defect) on a line with coorientation means reverse orientation of σ.



Now, we can build a network on a plane.



network

So if we have a point p in the network $\mathcal{N}$, we can define the winding number:

$\omega(p, \mathcal{N}) =$ take a path from p to infinity, then read the labels of intersection points.

If coorientation points away from p (outwards direction), put σ.

If coorientation points towards p (inward direction), put σ^{-1}.

$\omega(p, \mathcal{N})$ is an element of G.

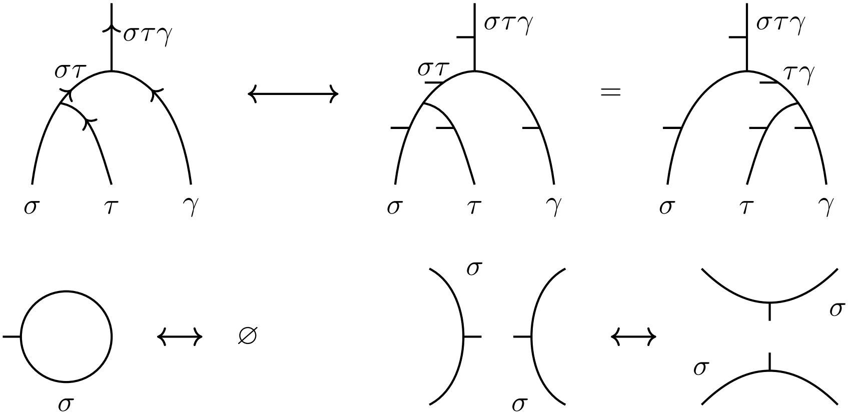
Follow along **orange** dotted line to obtain $\omega(p, \mathcal{N}) = \sigma\tau \in G$.

The winding number doesn't depend on the choice of a path.

Follow along **red** dotted line to obtain $\omega(p, \mathcal{N}) = (\sigma\tau\gamma)\gamma^{-1} = \sigma\tau \in G$.

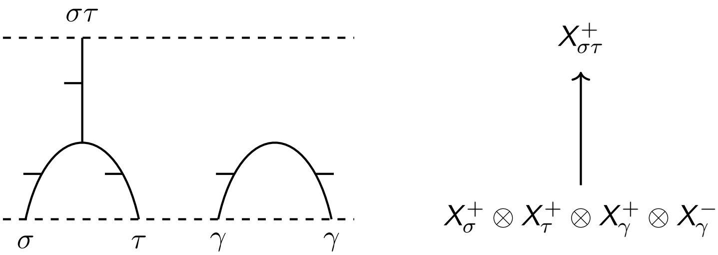
We can build a monoidal category. But first, need equivalence relation.

Some of the relations:



Closed loop vanishes, and we allow local operation (surgery) as on the right hand side.

Extend network to network with boundaries:



Now, we can build a category that is monoidal:

Take tensor product of objects by putting networks next to each other,

Compose morphisms by stacking one on top of another.

We can bend the network $\Rightarrow$ the category is pivotal. So for each object, we have a dual object.

For every closed network, it is cobordant to the empty set since the classifying space BG is 1. That is, a planar G-network (without dots) can be interpreted as a map to the classifying space BG by passing to the Poincaré dual of the network. A map of a sphere S^2 into BG is contractible since BG has $\pi_2(BG) = \pi_3(BG) = \pi_4(BG) = \ldots = 0$ and $\pi_1(BG) = G$.

There is very little freedom in this category. Given a product of corresponding objects from one to another, there is a ! morphism given by some network. In particular, $X_\sigma^+ \otimes X_\tau^+ \otimes X_\gamma^+ \otimes X_\gamma^- \simeq X_{\sigma\tau}^+$.

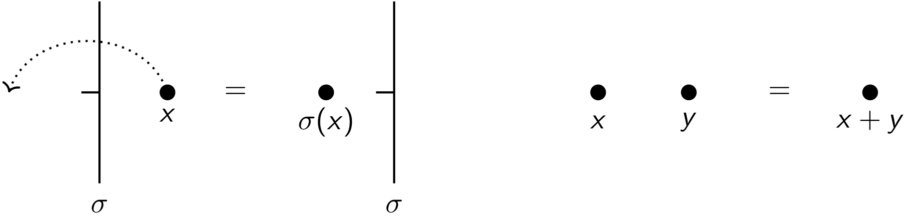
The dual picture is useful in physics, where we have defect lines that merge and split.

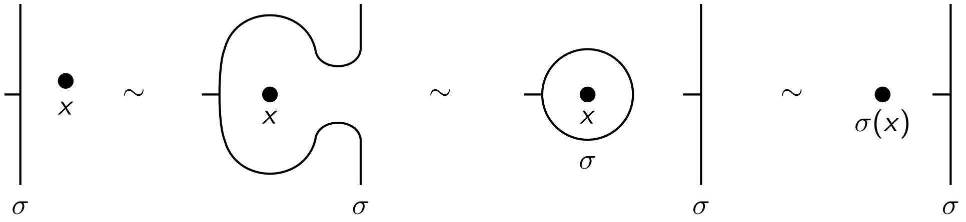
G-representation U.

Another modification we can do is add a representation U of the group G.

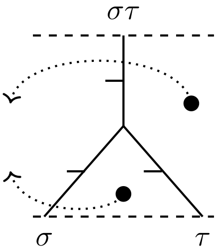
Left hand side: dots labelled elements of U can float in regions and move across $x \mapsto \sigma(x)$.

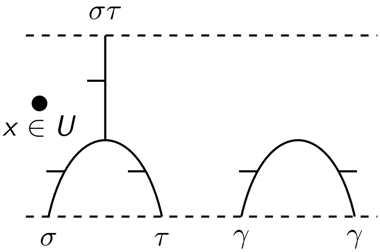
Right hand side: additivity of the floating dots.





So we can move all the dots to the left (far left as possible), by passing through the lines. This dot introduces a twist by the group element $\sigma \in G$. As a final step, we can combine all the dots into one dot.





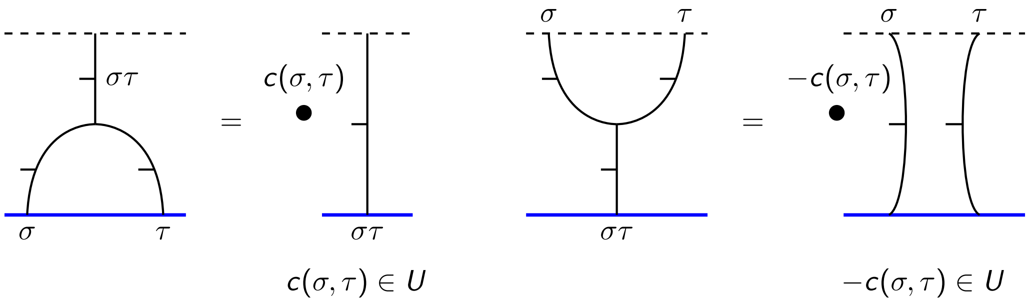
morphisms constitute a U-torsor

A torsor is a copy of U but without distinguished element (0). U acts freely and transitively on it.

If the top and bottom sequences, upon multiplication of labels at endpoints (with signs) give the same element of G, the space of homs is a copy of U and can be viewed as a U-torsor. If the sequences give different elements of G, the space of homs is empty (no homs).

We can modify in other ways.

Blue wall swallows an additive vertex in a network:

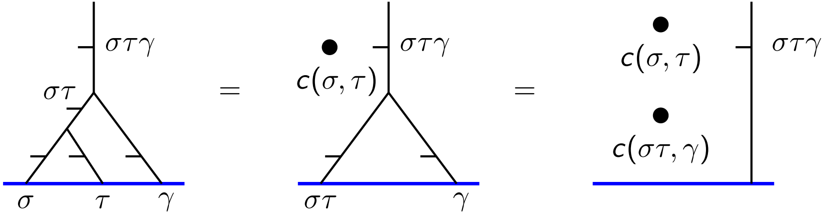


Have a wall, common in physics, that can swallow, absorb, or emit pieces of a network.

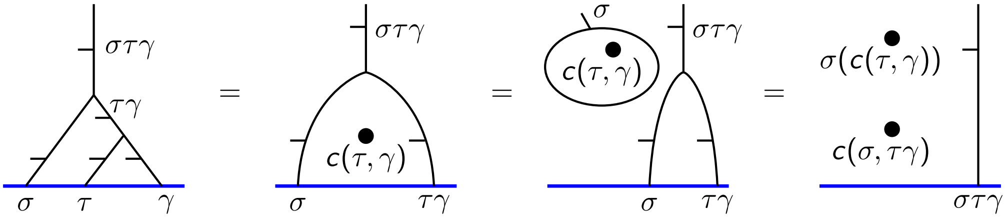
Left: a wall can absorb a vertex as well as 2 lines, and emit a dot with a label.

Right: a wall absorbs a vertex and a line segment, and emits a dot but with a negative sign since we have a split at the additive vertex.

What is the condition on this c?



|| associativity relation



We have

$$c(\sigma, \tau) + c(\sigma\tau, \gamma) = \sigma(c(\tau, \gamma)) + c(\sigma, \tau\gamma).$$

So σ is a 2-cocycle on G with value in the representation U. The action of G may be nontrivial, and that's why one of the terms has action of σ in the equality above.

This gives us another way to think about 2-cocycles: using absorptions and emissions of networks and vertices.

Original category can be modified using a 2-cocycle c. The new category is still monoidal.

Interpret Shannon's entropy of a finite random variable.

$X = \{x_1, \ldots, x_n\}$, a finite set.

Shannon's entropy of a probability distribution p_X on X that associates probabilities $p_1, \ldots, p_n$, $\sum_{i=1}^{n} p_i = 1$, $0 < p_i < 1$ to its points x_i, is given by

$$H(p_X) \ := \ -\sum_{i=1}^{n} p_i \log p_i. \tag{1}$$

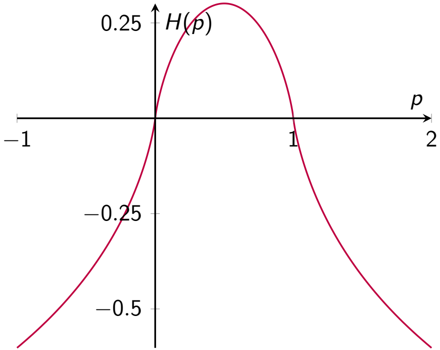
When $n = 2$, think of $H(p_X)$ as a function of a single probability $p = p_1$.
So the random variable is $(p, 1-p)$, $0 < p < 1$.
The entropy is

$$H(p) \ = \ -p \log p - (1-p) \log(1-p). \tag{2}$$

It is natural to extend H from $(0,1)$ to all real numbers by

$$H(p) := \begin{cases} -p\log|p| - (1-p)\log|1-p| & \text{if } p \neq 0,1, \\ 0 & \text{if } p = 0,1. \end{cases}$$

With this extension, we get continuous $H(p)$ over all $\mathbb{R}$.



Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$, where

$$\psi(a) = \begin{cases} -a \log |a| & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

The map ψ is a nonlinear derivation:

$$\psi(ab) = \psi(a)b + a\psi(b), \quad \text{but} \quad \psi(a+b) \neq \psi(a) + \psi(b).$$

If $\underline{a} = (a_1, \ldots, a_n)$, then

$$H(\underline{a}) := \sum_{i=1}^{n} \psi(a_i) - \psi\left(\sum_{i=1}^{n} a_i\right).$$

If $\sum_{i=1}^{n} a_i = 1$, then $\psi\left(\sum_{i=1}^{n} a_i\right) = 0$. So get entropy for $\underline{a}$.

Entropy function satisfies a four-term functional equation (L. Faddeev, 1956)

$$H(p) - H(q) + p\, H\left(\frac{q}{p}\right) + (1-p)H\left(\frac{1-q}{1-p}\right) = 0, \qquad p \neq 0, 1. \tag{3}$$

Equivalently (more symmetric in p and q),

$$H(p) + (1-p)H\left(\frac{q}{1-p}\right) = H(q) + (1-q)H\left(\frac{p}{1-q}\right), \qquad p, q \neq 1.$$

Equation (3) together with $H(1-p) = H(p)$ and continuity property uniquely determines H as a nonzero function $\mathbb{R} \longrightarrow \mathbb{R}$.

k: a field of characteristic 0.

J.-L. Cathelineau: k-vector space $J(\mathrm{k})$.

spanning set: $\langle a, b\rangle$, $a, b \in \mathrm{k}$.

relations:

$$\langle a, b\rangle = \langle b, a\rangle \text{ (symmetry)},$$
$$\langle ca, cb\rangle = \mathrm{c}\ \langle a, b\rangle \text{ (scaling, } c \in \mathrm{k}),$$
$$\langle a, b + c\rangle + \langle b, c\rangle = \langle a + b, c\rangle + \langle a, b\rangle \text{ (2-cocycle relation, } (\mathrm{k}, +) \text{ is a group)}.$$

These relations imply that $\langle a, 0\rangle = \langle 0, a\rangle = 0$ and, $\langle a, -a\rangle = 0$.

$J(\mathrm{k})$: vector space of infinitesimal dilogarithms. Often infinite-dimensional over k.

It was observed by Cathelineau and Kontsevich, independently (also see review by Tom Leinster), that they can interpret entropy $H(p_X)$ as a suitable 2-cocycle for the affine group of symmetries $ax + b$ of a line.

That is,

$$\langle a, b\rangle_H := \begin{cases} (a+b)H\left(\frac{a}{a+b}\right) & \text{if } a+b \neq 0, \\ 0 & \text{if } a+b = 0. \end{cases}$$

Extend entropy for 2-variables with this formula.

We also have $\langle a, b\rangle_H = \psi(a) + \psi(b) - \psi(a+b)$ (nonlinear in a, b).

Proposition. $\langle a, b\rangle_H$ satisfies Cathelineau's relations above (the ground field being $k = \mathbb{R}$).

We can scale by c:

$$\mathrm{Aff}_1(\mathbb{R}) = \left\{ \begin{pmatrix} c & a \\ 0 & 1 \end{pmatrix} : c \in \mathbb{R}^*, a \in \mathbb{R} \right\},$$

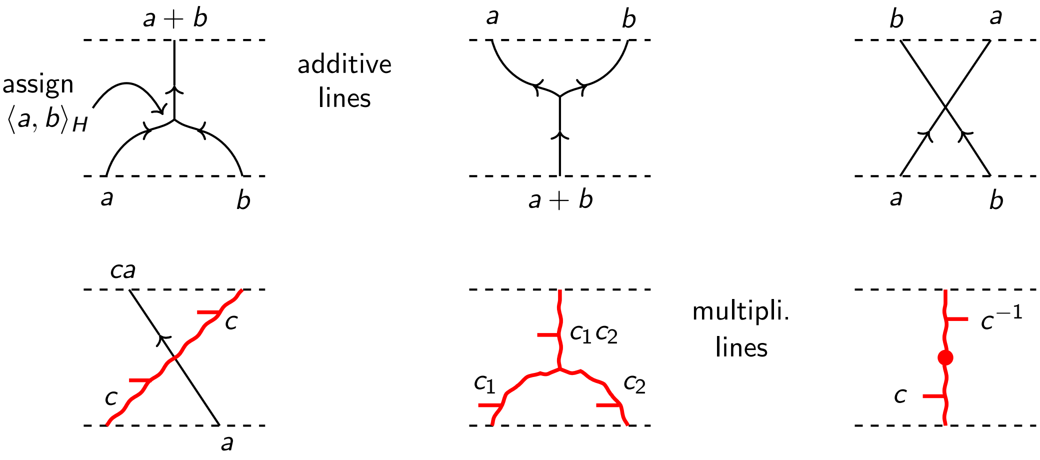
the group of affine symmetries of a line $x \mapsto cx + a, \quad c \neq 0,\ c, a \in \mathbb{R}$.

$$\mathrm{Aff}_1(\mathbb{R}) \quad \leftrightarrow \quad (c, a) \in \mathbb{R}^\times \times \mathbb{R} \quad \leftrightarrow \quad (a, c) \in \mathbb{R} \times \mathbb{R}^\times.$$

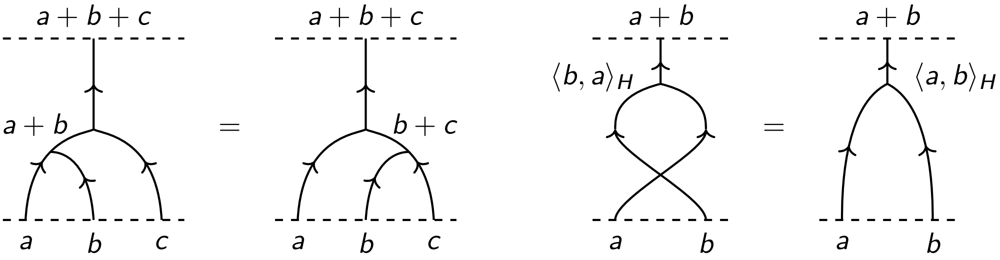
Extend $\langle\, , \,\rangle_H$ to a 2-cocycle on $\mathrm{Aff}_1(\mathbb{R})$:

$$\langle (a_1, c_1), (a_2, c_2) \rangle_H = \langle a_1, c_1 a_2 \rangle_H.$$

Graphical calculus of entropy.

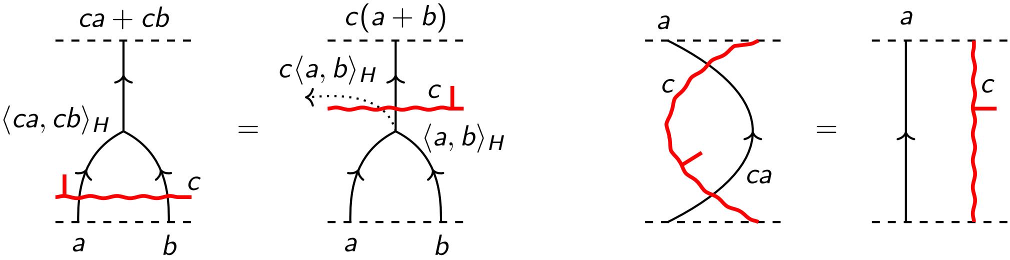


$c \otimes a \mapsto ca \otimes c$



Left: $\langle a, b\rangle_H + \langle a+b, c\rangle_H = \langle b, c\rangle_H + \langle a, b+c\rangle_H$.

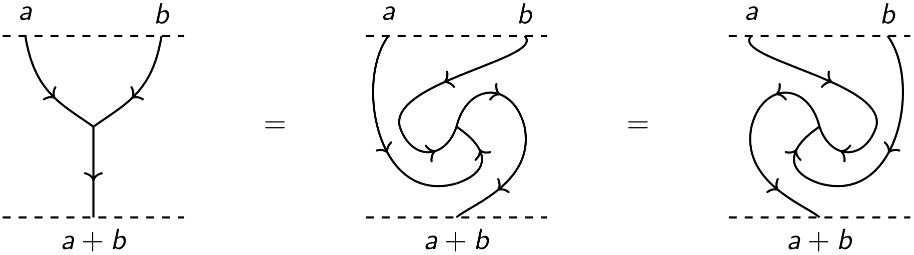
Right: $\langle b, a\rangle_H = \langle a, b\rangle_H$.



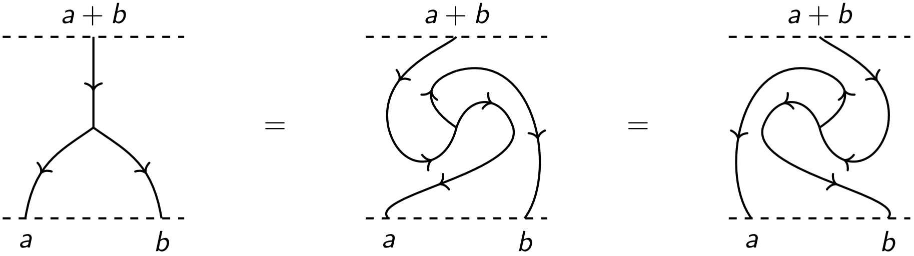
Push additive vertices to ∞.

Left: $\langle ca, cb\rangle_H = c\langle a, b\rangle_H$.

Right: virtual crossings give no contribution.

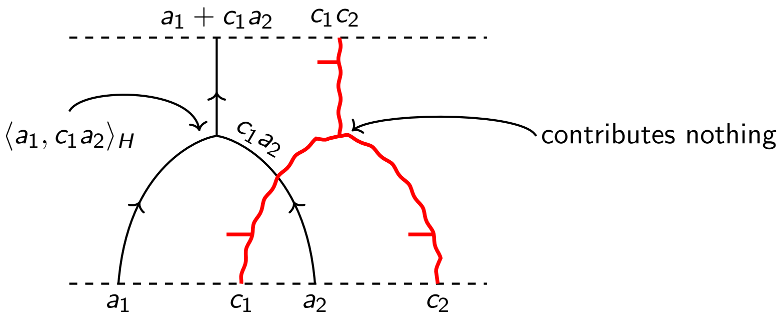


$\langle a, b\rangle_H$



$-\langle a, b\rangle_H$

So we get isotopy (diagrams can bend in all directions).



Labels of boundary points for this diagram match the multiplication rule in $\mathrm{Aff}_1(\mathbb{R})$: $(a_1, c_1)(a_2, c_2) = (a_1 + c_1 a_2, c_1 c_2)$. This diagram corresponds to multiplication in $\mathrm{Aff}_1(\mathbb{R})$.

Recall extending $\langle \ , \ \rangle_H$ to a 2-cocycle on $\mathrm{Aff}_1(\mathbb{R})$:

$$\langle (a_1, c_1), (a_2, c_2) \rangle_H := \langle a_1, c_1 a_2 \rangle_H .$$

Proposition. Two-cocycle $\langle \ , \ \rangle_H$ on $\mathrm{Aff}_1(\mathbb{R})$ is not a coboundary.

Note that the restriction of the 2-cocycle $\langle \ , \ \rangle_H$ to $\mathbb{R}$ is a coboundary, via the function ψ, since $\langle a, b \rangle_H = \psi(a) + \psi(b) - \psi(a + b)$.

k-vector space $\beta(\mathsf{k})$ has a spanning set $[a]$, $a \in \mathsf{k}^\times$, of vectors and relations:

$$[1] = 0,$$

$$[a] - [b] + a\left[\frac{b}{a}\right] + (1-a)\left[\frac{1-b}{1-a}\right] = 0, \quad a \in \mathsf{k} \setminus \{0,1\}, \quad b \in \mathsf{k}^\times.$$

Very similar to the 4-term equation:

$$H(p) - H(q) + p\,H\left(\frac{q}{p}\right) + (1-p)H\left(\frac{1-q}{1-p}\right) = 0, \quad p \neq 0, 1.$$

$\beta(\mathsf{k})$: generalizes entropy. This 4-term equation is a limit (deformation) of the 5-term equation for the dilogarithm (see [Zagier]), hence the name *infinitesimal dilogarithm*.

So we can also view entropy as a function $\beta(\mathbb{R}) \xrightarrow{H} \mathbb{R}, \qquad [a] \mapsto H(a)$.

Conditions on H: only needs to be a continuous function.

$H : \mathbb{R} \longrightarrow \mathbb{R}$, unique continuous function that respects the 4-term relation above.

Cathelineau proves that there is an isomorphism of k-vector spaces

$$R : \beta(\mathrm{k}) \xrightarrow{\cong} J(\mathrm{k}), \quad [a] \longmapsto \langle a, 1-a\rangle\,, \tag{4}$$

with the inverse map R^{-1} given by

$$\langle a, -a\rangle \longmapsto 0, \quad \langle a, b\rangle \longmapsto (a+b)\left[\frac{a}{a+b}\right] \quad \text{if} \;\; a+b \neq 0. \tag{5}$$

Symbols $\langle a, b\rangle$ are reminiscent of values of 2-cocycles, and Cathelineau shows that $H_2(\mathrm{Aff}_1(\mathrm{k}), \mathrm{k}_r) \simeq \beta(\mathrm{k}) \simeq J(\mathrm{k})$, where k_r is a suitable right $\mathrm{Aff}_1(\mathrm{k})$-module.

Here, $cx + d \in \mathrm{Aff}_1(\mathrm{k})$ acts on $y \in k_r \cong k$ by multiplication by c^{-1}.

Ω^1_{k}: the space of degree one absolute Kähler differentials.

If $\mathrm{char}(\mathrm{k}) = 0$, the following sequence of k-vector spaces is exact:

$$0 \longrightarrow \beta(\mathrm{k}) \xrightarrow{f} \mathrm{k}^{+} \otimes \mathrm{k}^{\times} \xrightarrow{g} \Omega^1_{\mathrm{k}} \longrightarrow 0. \tag{6}$$

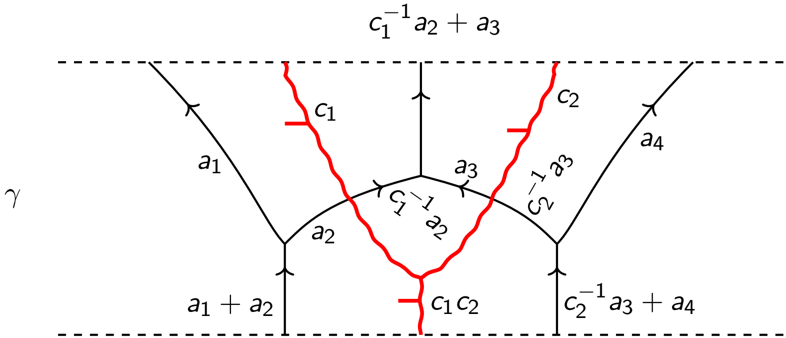
If $\mathrm{char}(\mathrm{k}) = p > 0$, the following sequence of k-vector spaces is exact:

$$0 \longrightarrow \mathrm{k} \xrightarrow{h} \beta(\mathrm{k}) \xrightarrow{f} \mathrm{k}^{+} \otimes \mathrm{k}^{\times} \xrightarrow{g} \Omega^1_{\mathrm{k}} \longrightarrow 0. \tag{7}$$

In these sequences the maps are

$$f([a]) = a \otimes a + (1-a) \otimes (1-a), \quad \text{for } a \neq 0, 1; \quad f([0]) = 0,$$
$$g(b \otimes a) = b\frac{da}{a} = b\, d\log(a), \quad h(1) = \sum_{c \in \mathrm{k}_0} [c],$$

where $\mathrm{k}_0 = F_p$, a prime subfield of k.



$$\jmath(\gamma) = -\langle a_1, a_2\rangle + c_1\langle c_1^{-1}a_2, a_3\rangle - c_1 c_2\langle c_2^{-1}a_3, a_4\rangle$$
$$= -\langle a_1, a_2\rangle + \langle a_2, c_1 a_3\rangle - \langle c_1 a_3, c_1 c_2 a_4\rangle$$

To a morphism γ, assign element $\jmath(\gamma)$ of $J(\mathrm{k})$. Generating objects of this monoidal category are labelled by $a \in \mathrm{k}$ (endpoints of additive networks) and $c \in \mathrm{k}^\times$ (endpoints of multiplicative networks).

Winding number.

Summation of vertices of additive network with action by $\mathrm{k}^\times$ through winding number.

Define

$$\jmath(\gamma) = \sum_{p \in \mathrm{add}(\gamma)} s(p)\omega(p,\gamma)\langle a_p, b_p\rangle \ \in \ J(\mathrm{k}). \tag{8}$$

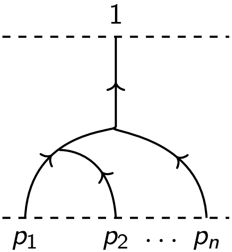
As an example, consider the morphism γ on previous slide, with three additive vertices, which we label p_1, p_2, p_3 from left to right. The leftmost point p_1 is a split, so $s(p_1) = -1$, and has $\mathrm{k}^\times$-winding number 1, since it is already in the leftmost region of the wavy red network. It contributes $-\langle a_1, a_2\rangle$ to $\jmath(\gamma)$. Point p_2 is a merge, $s(p_2) = 1$, the winding number $\omega(p_2,\gamma) = c_1$, and it contributes $c_1\langle c_1^{-1}a_2, a_3\rangle = \langle a_2, c_1 a_3\rangle$ to $\jmath(\gamma)$. Finally, point p_3 has $s(p_3) = -1$ and $\omega(p_3,\gamma) = c_1 c_2$.

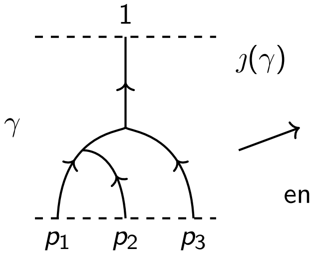
The invariant $\jmath(\gamma)$ is written on the previous slide.

Proposition. $\jmath(\gamma)$ depends only on the source and target objects of morphism γ, where

$$\jmath(\gamma) = \sum_{p \in \mathrm{add}(\gamma)} s(p)\omega(p,\gamma)\langle a_p, b_p\rangle \ \in \ J(\mathsf{k}).$$

Over $\mathbb{R}$, we recover entropy of random variables.





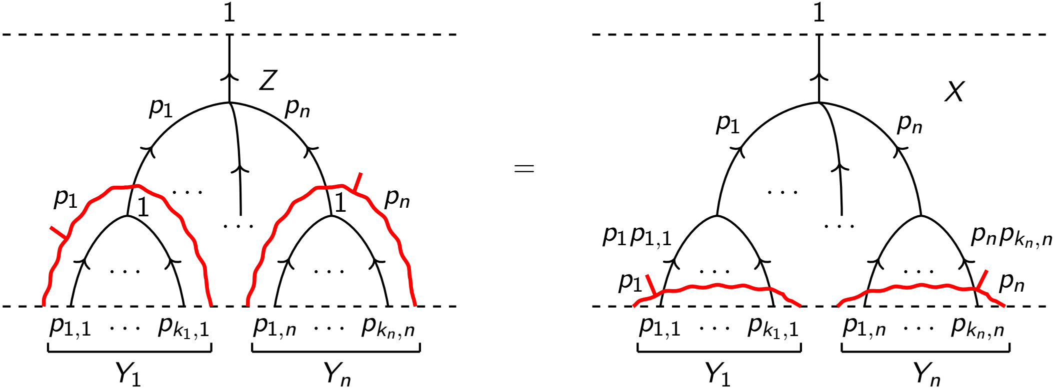
Element of $J(\mathbb{R})$. Apply H to get entropy of (p_1, p_2, p_3).

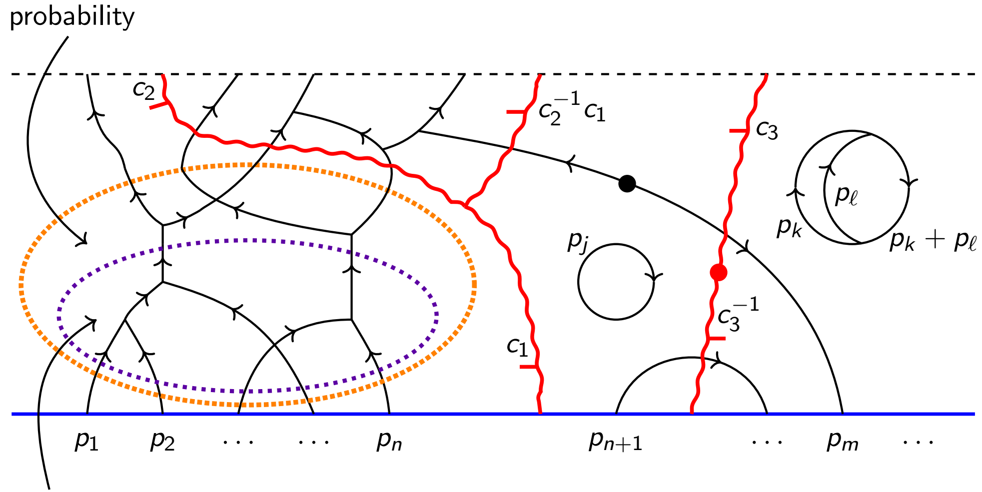
$$\jmath(\gamma) = \langle p_1, p_2\rangle + \langle p_1 + p_2, p_3\rangle + \dots$$
$$\dots + \langle 1 - p_n, p_n\rangle \xrightarrow{H \circ R^{-1}} H(\underline{p})$$

$$J(\mathbb{R}) \xrightarrow{R^{-1}} \beta(\mathbb{R}) \xrightarrow{H} \mathbb{R}$$

We also have a diagrammatic version of the chain rule:

$$H(X) = H(Z) + \sum_{i=1}^{n} p_i H(Y_i). \tag{9}$$





probability for
classical physics $\sum_{i=1}^{n} p_i = 1$

J. Baez, T. Leinster, and T.-D. Bradley have considered categorical interpretation of entropy of a finite random variable. There are similarities with their work in our approach.

In our case, the monoidal category contains a different kind of objects and morphisms, corresponding to multiplicative (red) lines that scale additive "probability" lines.
J.-L. Cathelineau's invariants also fit into the framework of similar monoidal categories.

We have seen that the entropy is related to cohomology and cocycles. One of the novel parts is the diagrammatic interpretation of that relation and the appearance of a monoidal category of planar diagrams that controls the cocycles.

As we have been above, a similar diagrammatic presentation is possible for general 2-cocycles in groups (and other low-degree cocycles) for a module U over a group G via planar diagrams. Elements of U correspond to floating points the plane, lines in a planar network are labelled by elements of G and an additive vertex on a network contributes the value of the 2-cocycle on a pair of group elements.

That is, let U be a $\mathbb{Z}[G]$-module, and assume given a 2-cocycle $c : G \times G \longrightarrow U$, so that

$$\sigma(c(\tau,\gamma)) + c(\sigma,\tau\gamma) = c(\sigma,\tau) + c(\sigma\tau,\gamma), \quad \sigma,\tau,\gamma \in G. \tag{10}$$

Depict elements of U by dots floating in the plane and build a planar network where co-oriented edges are labelled by elements of G and lines σ, τ can merge into the line $\sigma\tau$ and vice versa. Introduce a wall that can absorb and emit vertices, subject to the rules as discussed before.

So the 2-cocycle relation can be interpreted as the associativity relation on merging 3 strands σ, τ, γ into $\sigma\tau\gamma$.

Proof 1.

A: a unital commutative ring.

P_A: an abelian group generated by $\{z\}$, where z and $1-z$ are invertible in A, satisfying relations

$$\{z\} + \left\{\frac{1}{z}\right\} = 0,$$

$$\left\{\frac{1}{z}\right\} - \{1-z\} = 0,$$

$$\{z_1\} - \{z_2\} + \left\{\frac{z_2}{z_1}\right\} - \left\{\frac{1-z_2}{1-z_1}\right\} + \left\{\frac{(1-z_2)z_1}{(1-z_1)z_2}\right\} = 0,$$

where the last relation is called the 5-term *dilogarithm* equation.

A homomorphism of unital rings $A \to A'$ induces a homomorphism of abelian groups $P_A \to P_{A'}$.

Specialize to the case when $A = \mathsf{k}[\varepsilon]$, $\varepsilon^2 = 0$, and $A' = \mathsf{k}$.

Let $\widetilde{T}P(\mathsf{k}) := \ker(P_{\mathsf{k}[\varepsilon]} \to P_{\mathsf{k}})$, where the homomorphism $\mathsf{k}[\varepsilon] \longrightarrow \mathsf{k}$ is identity on k and $\varepsilon \mapsto 0$.

This induces the short exact sequence

$$0 \longrightarrow \widetilde{T}P(\mathsf{k}) \hookrightarrow P_{\mathsf{k}[\varepsilon]} \twoheadrightarrow P_{\mathsf{k}} \longrightarrow 0, \qquad \sigma: P_{\mathsf{k}[\varepsilon]} \to \widetilde{T}P(\mathsf{k})$$

with the section σ given by

$$\sigma(\{a + b\varepsilon\}) = \{a + b\varepsilon\} - \{a\}, \qquad a \neq 0, 1. \tag{11}$$

Kernel $\widetilde{T}P(\mathrm{k})$ is also generated by the images of $\{a+b\varepsilon\}$ under σ as in (11) over all $a \in \mathrm{k}^* \setminus \{1\}$ and $b \in \mathrm{k}$.

Also have k^* acting on $\widetilde{T}P(\mathrm{k})$ via the translation on the coefficient of ε:

$$c\,\sigma(\{a+b\varepsilon\}) = c(\{a+b\varepsilon\} - \{a\}) = \{a+cb\varepsilon\} - \{a\} = \sigma(\{a+cb\varepsilon\}).$$

Let $\mu : \widetilde{T}P(\mathrm{k}) \to \mathrm{k}^+ \wedge_{\mathbb{Z}} \mathrm{k}^+$ be a morphism of abelian groups, where $\mu(\{a+b\varepsilon\} - \{a\}) = \frac{b}{a} \wedge \frac{b}{1-a}$, or alternatively,
$\mu(\{a+2b\varepsilon\} + \{a\} - 2\{a+b\varepsilon\}) = 2\left(\frac{b}{a} \wedge \frac{b}{1-a}\right) \neq 0$.

Let N_{k} be the subgroup of $\widetilde{T}P(\mathrm{k})$ generated by

$$\{a+(b+b')\varepsilon\} + \{a\} - \{a+b\varepsilon\} - \{a+b'\varepsilon\}. \qquad (12)$$

In $\widetilde{T}P(\mathrm{k})$, view $\{a+b\varepsilon\} - \{a\}$ as equal to its deformation. That is, let $TP(\mathrm{k}) := \widetilde{T}P(\mathrm{k})/N_{\mathrm{k}}$. Then in $TP(\mathrm{k})$, we have

$$\{a+(b+b')\varepsilon\} - \{a+b'\varepsilon\} = \{a+b\varepsilon\} - \{a\}.$$

Proposition. For k a field of characteristic 0, there exists an $\cong$ of vector spaces

$$\varphi : \beta_{\mathrm{k}} \xrightarrow{\cong} TP(\mathrm{k}), \qquad \text{where } \varphi([a]) = \{a + a(1-a)\varepsilon\} - \{a\}. \tag{13}$$

This leads us to a commutative diagram with two short exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & N_{\mathrm{k}} & & & & & & \\
 & & \big\downarrow & & \overset{\sigma}{\curvearrowleft} & & & & \\
0 & \longrightarrow & \widetilde{T}P(\mathrm{k}) & \hookrightarrow & P_{\mathrm{k}[\varepsilon]} & \twoheadrightarrow & P_{\mathrm{k}} & \longrightarrow & 0 \\
 & & \Big\downarrow \text{quotient by } N_{\mathrm{k}} & & & & & & \\
\beta_{\mathrm{k}} & \xrightarrow[\cong]{\varphi} & TP(\mathrm{k}) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0. & & & & & &
\end{array}$$

Proof. We will prove that φ is well-defined. Let

$$u = a + a(1-a)\varepsilon, \qquad v = b + b(1-b)\varepsilon.$$

Then

$$\begin{aligned}
\frac{u(1-v)}{v(1-u)} &= \frac{(a + a(1-a)\varepsilon)(1-(b+b(1-b)\varepsilon))}{(b+b(1-b)\varepsilon)(1-(a+a(1-a)\varepsilon))} \\
&= \frac{a - ab + a(1-b)(1-a-b)\varepsilon}{b - ab + b(1-a)(1-a-b)\varepsilon} \\
&= \frac{a\cancel{b}\cancel{(1-a)}(1-b)}{b^{\cancel{2}}(1-a)^{\cancel{2}}} \\
&= \frac{a(1-b)}{b(1-a)}.
\end{aligned}$$

$$
\begin{aligned}
&\varphi\left([a]-[b]+(1-a)\left[\frac{1-b}{1-a}\right]+a\left[\frac{b}{a}\right]\right)\\
&=\varphi([a])-\varphi([b])+(1-a)\varphi\left(\left[\frac{1-b}{1-a}\right]\right)+a\varphi\left(\left[\frac{b}{a}\right]\right)\\
&=\{a+a(1-a)\varepsilon\}-\{a\}-(\{b+b(1-b)\varepsilon\}-\{b\})\\
&\quad+(1-a)\left(\left\{\frac{1-b}{1-a}+\frac{1-b}{1-a}\left(1-\frac{1-b}{1-a}\right)\varepsilon\right\}-\left\{\frac{1-b}{1-a}\right\}\right)\\
&\quad+a\left(\left\{\frac{b}{a}+\frac{b}{a}\left(1-\frac{b}{a}\right)\varepsilon\right\}-\left\{\frac{b}{a}\right\}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma(\{a + a(1-a)\varepsilon\}) - \sigma(\{b + b(1-b)\varepsilon\}) \\
&\quad + (1-a)\sigma\left(\left\{\frac{1-b}{1-a} + \frac{1-b}{1-a}\left(1 - \frac{1-b}{1-a}\right)\varepsilon\right\}\right) + a\sigma\left(\left\{\frac{b}{a} + \frac{b}{a}\left(1-\frac{b}{a}\right)\varepsilon\right\}\right) \\
&= \sigma(\{u\}) - \sigma(\{v\}) + \sigma\left(\left\{\frac{1-b}{1-a} + (1-b)\left(1 - \frac{1-b}{1-a}\right)\varepsilon\right\}\right) \\
&\quad + \sigma\left(\left\{\frac{b}{a} + b\left(1-\frac{b}{a}\right)\varepsilon\right\}\right) \\
&= \sigma(\{u\}) - \sigma(\{v\}) + \sigma\left(\left\{\frac{1-b}{1-a} - \frac{1-b}{1-a}(a-b)\varepsilon\right\}\right) + \sigma\left(\left\{\frac{b}{a} + \frac{b}{a}(a-b)\varepsilon\right\}\right) \\
&\overset{\clubsuit}{=} \sigma(\{u\}) - \sigma(\{v\}) + \sigma\left(\left\{\frac{b}{a} + \frac{b}{a}(a-b)\varepsilon\right\}\right) - \sigma\left(\left\{\frac{1-b}{1-a} + \frac{1-b}{1-a}(a-b)\varepsilon\right\}\right) \\
&\quad + \left\{\frac{a(1-b)}{b(1-a)}\right\} - \left\{\frac{a(1-b)}{b(1-a)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma(\{u\}) - \sigma(\{v\}) + \sigma\left(\left\{\frac{b}{a} + \frac{b}{a}(a-b)\varepsilon\right\}\right) - \sigma\left(\left\{\frac{1-b}{1-a} + \frac{1-b}{1-a}(a-b)\varepsilon\right\}\right) \\
&\quad + \sigma\left(\left\{\frac{a(1-b)}{b(1-a)}\right\}\right) \\
&= \sigma\left(\{u\} - \{v\} + \left\{\frac{b}{a}(1+(a-b)\varepsilon)\right\} - \left\{\frac{(1-b)+(1-b)(a-b)\varepsilon}{1-a}\right\} + \left\{\frac{a(1-b)}{b(1-a)}\right\}\right) \\
&= \sigma\left(\{u\} - \{v\} + \left\{\frac{v}{u}\right\} - \left\{\frac{1-v}{1-u}\right\} + \left\{\frac{u(1-v)}{v(1-u)}\right\}\right)
\end{aligned}
$$

since

$$\frac{v}{u} = \frac{b+b(1-b)\varepsilon}{a+a(1-a)\varepsilon} = \frac{b+b(1-b)\varepsilon}{a+a(1-a)\varepsilon}\frac{a-a(1-a)\varepsilon}{a-a(1-a)\varepsilon}$$
$$= \frac{ab+ab(a-b)\varepsilon}{a^2} = \frac{b}{a} + \frac{b}{a}(a-b)\varepsilon,$$

$$\frac{1-v}{1-u} = \frac{1-(b+b(1-b)\varepsilon)}{1-(a+a(1-a)\varepsilon)} = \frac{1-b-b(1-b)\varepsilon}{1-a-a(1-a)\varepsilon}$$
$$= \frac{1-b-b(1-b)\varepsilon}{1-a-a(1-a)\varepsilon}\frac{1-a+a(1-a)\varepsilon}{1-a+a(1-a)\varepsilon}$$
$$= \frac{(1-a)(1-b)+(1-a)(1-b)(a-b)\varepsilon}{(1-a)^2}$$
$$= \frac{(1-b)+(1-b)(a-b)\varepsilon}{1-a}$$
$$= \frac{1-b}{1-a} + \frac{1-b}{1-a}(a-b)\varepsilon,$$

$$
\begin{aligned}
\frac{u(1-v)}{v(1-u)} &= \frac{(a+a(1-a)\varepsilon)(1-(b+b(1-b)\varepsilon))}{(b+b(1-b)\varepsilon)(1-(a+a(1-a)\varepsilon)} \\
&= \frac{a(1-b)\cancel{(1+(1-(a+b))\varepsilon)}}{b(1-a)\cancel{(1+(1-(a+b))\varepsilon)}} \\
&= \frac{a(1-b)}{b(1-a)}.
\end{aligned}
$$

And for $b' = -b$ in (12), we have

$$
\{a+(b-b)\varepsilon\} + \{a\} - \{a+b\varepsilon\} - \{a-b\varepsilon\} = \{a\} + \{a\} - \{a+b\varepsilon\} - \{a-b\varepsilon\}.
$$

So in $TP(\mathrm{k})$,

$$\begin{aligned} 0 &= \sigma(\{a\} + \{a\} - \{a + b\varepsilon\} - \{a - b\varepsilon\}) \\ &= 2\sigma(\{a\}) - \sigma(\{a + b\varepsilon\}) - \sigma(\{a - b\varepsilon\}) \\ &= 2(\{a\} - \{a\}) - \sigma(\{a + b\varepsilon\}) - \sigma(\{a - b\varepsilon\}) \\ &= 0 - \sigma(\{a + b\varepsilon\}) - \sigma(\{a - b\varepsilon\}). \end{aligned}$$

The equality ♣ holds in $TP(\mathrm{k})$ since

$$-\sigma(\{a + b\varepsilon\}) = \sigma(\{a - b\varepsilon\}).$$

Now to prove that φ is injective, let

$$\rho : TP(\mathsf{k}) \longrightarrow \mathsf{k}^* \otimes \mathsf{k}^+, \qquad \rho(\{a + b\varepsilon\} - \{a\}) = a \otimes \frac{b}{1-a} + (1-a) \otimes \frac{b}{a}, \qquad (14)$$

be a map of vector spaces over k. Then we have a commutative diagram

$$\begin{array}{ccc} \beta_{\mathsf{k}} & \xhookrightarrow{\quad D \quad} & \mathsf{k}^* \otimes \mathsf{k}^+, \\ & {\scriptstyle \varphi} \searrow \quad \swarrow {\scriptstyle \rho} & \\ & TP(\mathsf{k}) & \end{array}$$

where D and ρ are injective. So φ is also injective.

■

Proof 2.

Let k be a field, and let $k^\flat = k^\times \setminus \{1\} = k \setminus \{0,1\}$.

Let $B_2(k)$ be the quotient of $\mathbb{Q}[k^\flat]$ by

$$[a] - [b] + \left[\frac{b}{a}\right] - \left[\frac{1-a^{-1}}{1-b^{-1}}\right] + \left[\frac{1-a}{1-b}\right] = 0,$$

with $(1-a)(1-b)\left(1-\frac{a}{b}\right) \in k^\times$.

Let $k_2 := k[t]/(t^2)$ be the ring of dual numbers.

If $a \in k^\flat$, define

$$\langle a \rangle := a + a(1-a)t = a(1+(1-a)t) \in k_2.$$

Note that $(1+ct)^{-1} = 1 - ct$ in k_2 since $t^2 = 0$. We see that $\langle a \rangle$ is invertible in k_2, with $\langle a \rangle^{-1} = a^{-1}(1-(1-a)t)$.

Define the action of $k^\times$ on k_2 as $\lambda * (b_0 + b_1 t) = b_0 + \lambda b_1 t$, where $\lambda \in k^\times$. That is, $\lambda * 1 = 1$ and $\lambda * t = \lambda t$. So λ only translates the coefficient of t.

Lemma

We have $\dfrac{\langle b \rangle}{\langle a \rangle} = a * \left\langle \dfrac{b}{a} \right\rangle$.

Proof. The left hand side shows

$$\frac{\langle b\rangle}{\langle a\rangle}=\frac{b(1+(1-b)t)}{a(1+(1-a)t)}=\frac{b}{a}(1+((1-b)-(1-a))t)=\frac{b}{a}(1+(a-b)t).$$

On the other hand, the right hand side shows

$$\begin{aligned}a*\left\langle\frac{b}{a}\right\rangle&=a*\left(\frac{b}{a}\left(1+\left(1-\frac{b}{a}\right)t\right)\right)=\frac{b}{a}\left(1+\left(1-\frac{b}{a}\right)at\right)\\&=\frac{b}{a}\left(1+\frac{a-b}{a}at\right)=\frac{b}{a}(1+(a-b)t).\end{aligned}$$

This proves the lemma. □

Lemma

The relation

$$\frac{1-\langle a\rangle}{1-\langle b\rangle} = (b-1) * \left\langle \frac{1-a}{1-b} \right\rangle .$$

holds.

Proof. On the left hand side, we have

$$\frac{1-\langle a\rangle}{1-\langle b\rangle} = \frac{1-a(1+(1-a)t)}{1-b(1+(1-b)t)}$$

while on the right hand side, we have

$$\begin{aligned}(b-1)*\left\langle\frac{1-a}{1-b}\right\rangle &= (b-1)*\frac{1-a}{1-b}\left(1+\left(1-\frac{1-a}{1-b}\right)t\right)\\ &= \frac{1-a}{1-b}\left(1+\left(1-\frac{1-a}{1-b}\right)(b-1)t\right),\end{aligned}$$

which are indeed equal. □

Lemma

We have $1 - \langle a \rangle^{-1} = (1 - a^{-1})(1 - t)$.

Proof. Since

$$\begin{aligned} 1 - \langle a \rangle^{-1} &= 1 - a^{-1}(1 + (1 - a)t)^{-1} = 1 - a^{-1}(1 - (1 - a)t) \\ &= 1 - a^{-1} + a^{-1}(1 - a)t = 1 - a^{-1} + (a^{-1} - 1)t \\ &= (1 - a^{-1})(1 - t), \end{aligned}$$

the lemma holds. □

Lemma

The equation

$$\frac{1-\langle a\rangle^{-1}}{1-\langle b\rangle^{-1}} = \frac{1-a^{-1}}{1-b^{-1}}$$

holds.

The lemma is immediate since

$$\frac{1-\langle a\rangle^{-1}}{1-\langle b\rangle^{-1}} = \frac{(1-a^{-1})\cancel{(1-t)}}{(1-b^{-1})\cancel{(1-t)}} = \frac{1-a^{-1}}{1-b^{-1}}.$$

Lemma

We have $[b * \langle c\rangle] = b[\langle c\rangle]$ *where* $b \in \mathrm{k}$ *and* $c \in \mathrm{k}_2$.
Furthermore, $(-1) * [\langle 1-a\rangle] = -[\langle a\rangle]$ *and* $a * [\langle a^{-1}\rangle] = -[\langle a\rangle]$.

Lemma

The 5-term dilogarithm

$$[\langle a\rangle] - [\langle b\rangle] + \left[\frac{\langle b\rangle}{\langle a\rangle}\right] - \left[\frac{1-\langle a\rangle^{-1}}{1-\langle b\rangle^{-1}}\right] + \left[\frac{1-\langle a\rangle}{1-\langle b\rangle}\right] = 0 \qquad (15)$$

holds in the commutative ring $B_2(\mathrm{k}_2)$.

Theorem

Taking the limit $t \to 0$, the 5-term dilogarithm

$$[\langle a\rangle] - [\langle b\rangle] + \left[\frac{\langle b\rangle}{\langle a\rangle}\right] - \left[\frac{1-\langle a\rangle^{-1}}{1-\langle b\rangle^{-1}}\right] + \left[\frac{1-\langle a\rangle}{1-\langle b\rangle}\right] = 0$$

deforms to the 4-term infinitesimal dilogarithm

$$[a] - [b] + a\left[\frac{b}{a}\right] + (1-a)\left[\frac{1-b}{1-a}\right] = 0, \quad a \in \mathsf{k} \setminus \{0,1\},\ b \in \mathsf{k}^{\times}. \tag{16}$$

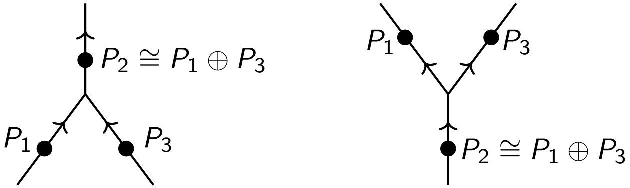
Algebraic K-theory and foam cobordisms.

Joint with David Gepner, Mikhail Khovanov, and Nitu Kitchloo.

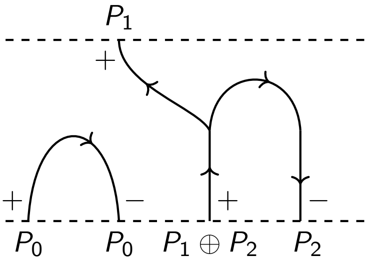
Consider oriented 1-foams as before, but now decorate them by fin. gen. projective modules P over a fixed ring R.

Along an edge, we have a flat connection with fiber a projective module P.

At trivalent vertices of 1-foams, fix a direct sum decomposition.



Consider such foams with 0-dimensional boundary. On the boundary, there is a sequence of signs and projective modules.



$$[P_0] - [P_0] + [P_1 \oplus P_2] - [P_2] = [P_1]$$
in $K_0(R)$

$K_0(R)$ is the abelian group generated by symbols $[P]$ of f. g. projective R-modules, with relations $[P_1] = [P_2] + [P_3]$ if $P_1 \cong P_2 + P_3$.

If R is a field k, any proj module is free $P \cong \mathrm{k}^n$, some n, and $K_0(\mathrm{k}) \cong \mathbb{Z}$.

Any free module is projective but not always vice versa (depends on R).

Why is $K_0(\mathrm{k}) \cong \mathbb{Z}$?

To compute K_0 of a field k, consider projective f. g. modules over k.

Any such module is isomorphic to k^n, for a unique $n \geqslant 0$.

In the Grothendieck group, $[\mathrm{k}^{n+m}] = [\mathrm{k}^n] + [\mathrm{k}^m]$ so that $[\mathrm{k}^n] = n[\mathrm{k}]$.

We see that K_0 has a generator $[\mathrm{k}]$, the symbol of 1-dim vector space over k, and no relations on it.

So $K_0(\mathrm{k}) = \mathbb{Z}[\mathrm{k}]$, the free abelian group on the generator $[\mathrm{k}]$.

R-decorated 0-*foams* mean a finite sequence of points with orientations ($+$ and $-$ signs) and f. g. proj R-module at each point.

View a 1-foam with boundary as a cobordism between R-decorated 0-foams.

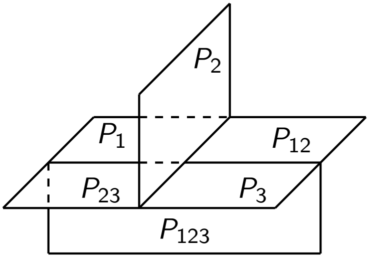
Top and bottom 0-foams define the same element of $K_0(R)$.

Theorem (GIKK). The Grothendieck group $K_0(R)$ of f. g. projective R-modules can be identified with the group of 0-dimensional decorated foams modulo cobordisms.

We think of our foams as carrying a flat connection with fibers - projective modules P. Can we do this in higher dimensions?

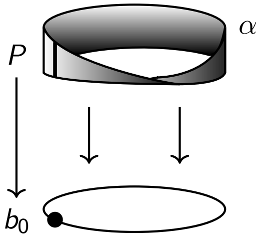
Let us go one dimension up and consider the group of 1-dimensional foams with such decorations modulo cobordisms. It is an abelian group with disjoint union as addition. We can bend a foam U to get $U^* = -U$.

Near a vertex of a 2-foam there are 6 facets and they carry decorations by projective modules P_1, P_2, P_3 and their sums $P_1 \oplus P_2$, $P_2 \oplus P_3$, $P_1 \oplus P_2 \oplus P_3$.



$$P_{12} \cong P_1 \oplus P_2,$$
$$P_{23} \cong P_2 \oplus P_3,$$
$$P_{123} \cong P_1 \oplus P_2 \oplus P_3.$$

A circle with a flat connection with fiber P is encoded by an automorphism $\alpha : P \longrightarrow P$, the monodromy of flat connection along the circle.



This is reminiscent of generators of the next K-theory group, $K_1(R)$. It is an abelian group generated by symbols $[(P, \alpha)]$ for P, α as above and relations

(a) $[(P, \alpha)] + [(P, \beta)] = [(P, \alpha\beta)]$,

(b) For a commutative diagram with short exact sequences in rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & P_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\ 0 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & P_3 & \longrightarrow & 0 \end{array}$$

and $\alpha_i \in \mathrm{Aut}(P_i)$,

$$[(P_2, \alpha_2)] \;=\; [(P_1, \alpha_1)] + [(P_3, \alpha_3)].$$

$K_1(R)$ is also isomorphic to the quotient group $\mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$, where $\mathrm{GL}(R)$ is the direct limit of $\mathrm{GL}_n(R)$ as $n \to \infty$. An element of $\mathrm{GL}_n(R)$ defines an automorphism of the free module R^n.

Example. For a field k, the group $K_1(\mathrm{k}) \cong \mathrm{k}^\times$. To (P, α), associate $\det(\alpha)$ (since any projective k-module is free).

Note: For any **commutative** ring R, an $n \times n$ matrix A with coefficients in R has determinant $\det(A)$ in R.

If A is invertible, $\det(A)$ is in $R^\times$, the group of invertible elements.

This gives a surjective map $K_1(R) \twoheadrightarrow R^\times$. Sometimes it is an isomorphism and sometimes it has a kernel.

When R is a field k, it is an isomorphism and thus $K_1(\mathrm{k}) \cong \mathrm{k}^\times$.

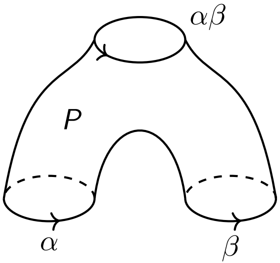
Note: if we have a projective module P and its automorphism α, how do we complete it to an element of $\mathrm{GL}_n(R)$?

We pick a module Q such that $P+Q$ isomorphic to R^n and extend α to an automorphism of $P+Q$ by combining with the identity automorphism of Q.

This gives us an automorphism of R^n, which is given by some invertible matrix, which defines an element of $\mathrm{GL}_n(R)$ and its quotient group $K_1(R)$.

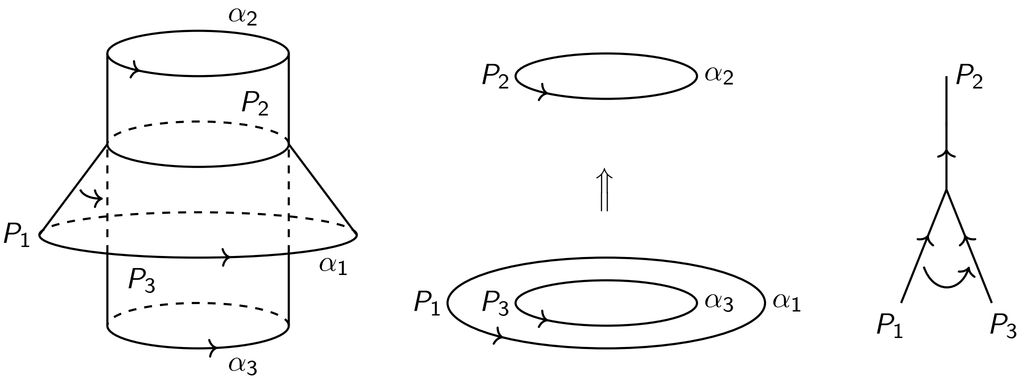
Consider 1-foams modulo 2-foam cobordisms. Both 1- and 2-foams carry decorations by projective R-modules. Then both relations above can be interpreted via cobordisms.

Relation (a):

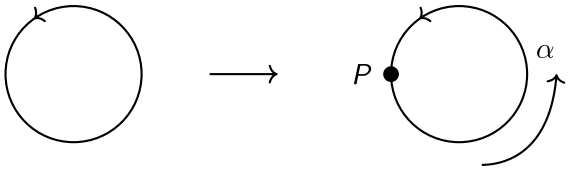


Over every point on this surface, there is a copy of P (flat bundle with fibers P). This copy changes as you travel along the surface, and α, β are the monodromies along the bottom circles. The monodromy along the top circle is $\alpha\beta$.

Relation (b):

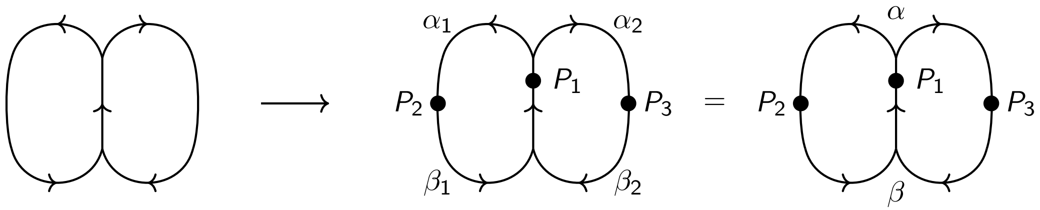

α_2
P_2
P_1
α_1
P_3
α_3
P_2
α_2
$\Uparrow$
P_1
P_3
α_3
α_1
P_2
P_1
P_3

Example. Put enough defects on a 1-foam to obtain elements of $K_1(R)$.



$$\begin{array}{cc} & P \\ P & \left[\ \alpha\ \right] \end{array}, \qquad \alpha \in \mathrm{Aut}(P)$$

Another Example. Put enough defects on a 1-foam to obtain elements of $K_1(R)$.



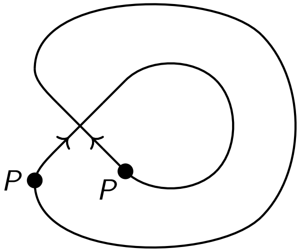
Our automorphism of $P_1 \oplus P_2 \oplus P_3$:

$$\begin{array}{c} \\ P_1 \\ P_2 \oplus P_3 \end{array} \begin{array}{c} \begin{array}{cc} P_1 & P_2 \oplus P_3 \end{array} \\ \left[\begin{array}{cc} 0 & (\beta_1, \beta_2) \\ (\alpha_1, \alpha_2) & 0 \end{array}\right] \end{array} = \begin{array}{c} \\ P_1 \\ P_2 \oplus P_3 \end{array} \begin{array}{c} \begin{array}{cc} P_1 & P_2 \oplus P_3 \end{array} \\ \left[\begin{array}{cc} 0 & \beta \\ \alpha & 0 \end{array}\right] \end{array}.$$

It turns out that the cobordism group of decorated 1-foams modulo cobordisms is almost $K_1(R)$.

It is $K_1(R)$ modulo a 2-torsion subgroup that comes from $K_0(R)$.

For each projective P, the transposition automorphism of $P \oplus P$ has order at most 2 in $K_1(R)$. This is represented by a circle with a crossing.



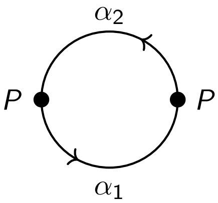
To clarify, the homomorphism τ takes a projective module P and assigns to it 2×2 matrix with rows and columns corresponding to P and entries

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrix induces an element of $K_1(R)$, the maximal abelian quotient of $\mathrm{GL}(R)$.

This construction gives a homomorphism $\tau : K_0(R) \longrightarrow K_1(R)$, and its image is a 2-torsion subgroup of R.

Example. In the figure below, we added an extra defect.



monodromy
$\alpha_2\alpha_1 = \alpha$

$$X = \begin{array}{c} \\ P \\ P \end{array} \begin{array}{cc} P & P \\ \hline \end{array} \begin{bmatrix} 0 & \alpha_2 \\ \alpha_1 & 0 \end{bmatrix}, \qquad \det(X) = -\alpha_2\alpha_1 = -\alpha$$

We obtain a sign change.

Since the minus sign appears, we need to mod out by $\operatorname{im}\tau = \tau(K_0(R))$.

Theorem (GIKK). The cobordism group of decorated 1-foams is isomorphic to $K_1(R)/\tau(K_0(R))$.

Example. For R a field k, $K_0(\mathrm{k}) \cong \mathbb{Z}$, the cobordism group is $\mathrm{k}^\times/\{\pm 1\}$.

Sometimes the subgroup $\tau(K_0(R))$ is just $\{1, -1\}$, but potentially it could be bigger. But it always consists of only 2-torsion elements.

Self-intersection of the circle 1-foam creates a problem for us and forces to quotient out by the image of τ.

Self-intersection of an edge carrying P creates a transposition of $P \oplus P$.

To avoid that, we restrict to 1-foams without self-intersections, that is, to 1-foams embedded into the plane.

To get exactly $K_1(R)$, we restrict to decorated 1-foams *embedded* in $\mathbb{R}^2$ and 2-foam cobordisms between them embedded in $\mathbb{R}^2 \times [0,1]$.

Theorem (GIKK). $K_1(R)$ is isomorphic to the cobordism group of 1-foams in the plane carrying a flat connection with fibers f. g. projective R-modules.

For the proof, we represent a network (decorated 1-foam) in the plane as the closure of a braid-like 1-foam. To a braid-like 1-foam we associate a projective module and its automorphism, establishing a map into $K_1(R)$ and later checking this is an isomorphism from the cobordism group onto $K_1(R)$.

Why interesting:

(a) Higher-dimensional foams and higher K-theory.

(b) Vary decorations on foams and cobordisms to get new flavors of K-theory.

Thank you!