A new construction of simple modules for type A KLR algebras

Joint work with Robert Muth, Thomas Nicewicz, and Louise Sutton

Liron Speyer

OIST Okinawa Institute of Science and Technology

Cuspidal systems for KLR algebras

- $e \in \mathbb{Z}_{>1}$: a fixed integer.
- $U_q(\widehat{\mathfrak{sl}}_e(\mathbb{C}))$: quantum group of type $\mathtt{A}^{(1)}_{e-1}$.
- $I = \{\alpha_0, \dots, \alpha_{e-1}\}$: the set of simple roots.
- $\delta = \alpha_0 + \cdots + \alpha_{e-1}$: the null root.
- Φ_+^{re}: the set of real roots.
- $\Phi_+^{\mathrm{im}} = \{d\delta \mid d \in \mathbb{Z}_{>0}\}$: the set of imaginary roots.
- $\Phi_+ = \Phi_+^{\mathrm{re}} \sqcup \Phi_+^{\mathrm{im}}$: the positive root system of type $\mathtt{A}^{(1)}_{e-1}$.
- $\Psi := \Phi_+^{\mathrm{re}} \sqcup \{\delta\}$: the set of indivisible roots.
- $\mathbb{F}$: an arbitrary field.
- $R(\omega)$: the type $\mathtt{A}^{(1)}_{e-1}$ KLR algebra over $\mathbb{F}$, for $\omega \in \mathbb{Z}_{\geqslant 0} I$. Generators $\{e(\mathbf{i}) \mid \mathbf{i} \in I^{\omega}\}$, $\{x_1, \dots, x_n\}$ $\{\psi_1, \dots, \psi_{n-1}\}$, $n := \mathrm{ht}(\omega)$. Many relations, but **not the cyclotomic one**. Their representation theory is studied via *cuspidal systems*, which are associated with PBW bases for the quantum group.

Cuspidal systems for KLR algebras

- $\succcurlyeq$: fixed convex preorder on Φ_+.
- A *Kostant partition* of $\omega \in \mathbb{Z}_{\geqslant 0}I$ is a tuple of non-negative integers $\boldsymbol{K} = (K_\beta)_{\beta\in\Psi}$ with $\sum_{\beta\in\Psi} K_\beta \beta = \omega$.
If $\beta_1 \succ \cdots \succ \beta_t$ are the elements of Ψ such that $K_{\beta_i} \neq 0$, then we write $\boldsymbol{K}$ in the form $\boldsymbol{K} = (\beta_1^{K_{\beta_1}} \mid \cdots \mid \beta_t^{K_{\beta_t}})$.
- A *root partition* of $\omega \in \mathbb{Z}_{\geqslant 0}I$ is a pair $\pi = (\boldsymbol{K}, \boldsymbol{\nu})$, where $\boldsymbol{K} = (\beta_1^{K_{\beta_1}} \mid \cdots \mid \beta_u^{K_{\beta_u}} \mid \delta^{K_\delta} \mid \beta_{u+1}^{K_{\beta_{u+1}}} \mid \cdots \mid \beta_t^{K_{\beta_t}})$ is a Kostant partition of ω and $\boldsymbol{\nu} = (\nu^{(1)} \mid \cdots \mid \nu^{(e-1)})$ is an $(e-1)$-multipartition of K_δ.
- $\Pi(\omega)$: the set of all root partitions of ω.

Cuspidal systems for KLR algebras

Definition

Let $m \in \mathbb{Z}_{>0}$, $\beta \in \Psi$. We say an $R(m\beta)$-module M is *semicuspidal* provided that for all $0 \neq \theta_1, \theta_2 \in \mathbb{Z}_{\geqslant 0} I$ with $\theta_1 + \theta_2 = m\beta$, we have $\mathrm{Res}^{R(m\beta)}_{R(\theta_1)\otimes R(\theta_2)} M \neq 0$ only if θ_1 is a sum of positive roots $\preccurlyeq \beta$ and θ_2 is a sum of positive roots $\succcurlyeq \beta$.

We say moreover that M is *cuspidal* if $m = 1$ and the comparisons above are strict.

Cuspidal and semicuspidal modules are key building blocks in the representation theory of $R(\omega)$.

To each $\beta \in \Phi_+^{\mathrm{re}}$, we associate a simple *cuspidal* $R(\beta)$-module $L(\beta)$, and to each $(e-1)$-multipartition $\boldsymbol{\nu}$ of $d \in \mathbb{Z}_{>0}$, we associate a simple *semicuspidal* $R(d\delta)$-module $L(\boldsymbol{\nu})$.

Cuspidal systems for KLR algebras

Then, to each $\pi = (\boldsymbol{K}, \boldsymbol{\nu}) \in \Pi(\omega)$, with $\boldsymbol{K} = (\beta_1^{K_{\beta_1}} \mid \cdots \mid \beta_u^{K_{\beta_u}} \mid \delta^{K_\delta} \mid \beta_{u+1}^{K_{\beta_{u+1}}} \mid \cdots \mid \beta_t^{K_{\beta_t}})$ and $\boldsymbol{\nu} = (\nu^{(1)} \mid \cdots \mid \nu^{(e-1)})$ an $(e-1)$-multipartition of K_δ, we associate the *proper standard module*

$$\bar{\Delta}(\pi) = L(\beta_1)^{\circ K_{\beta_1}} \circ \cdots \circ L(\beta_u)^{\circ K_{\beta_u}} \circ L(\boldsymbol{\nu}) \circ L(\beta_{u+1})^{\circ K_{\beta_{u+1}}} \circ \cdots \circ L(\beta_t)^{\circ K_{\beta_t}},$$

which has a self-dual simple head $L(\pi)$, and $\{L(\pi) \mid \pi \in \Pi(\omega)\}$ is a complete and irredundant set of simple $R(\omega)$-modules up to isomorphism and grading shift.

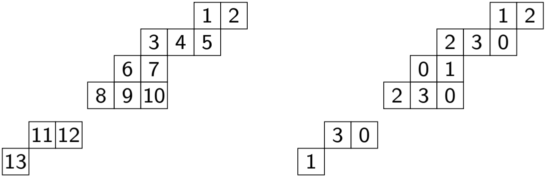
Previous constructions of simple semicuspidal modules $L(\boldsymbol{\nu})$ (and therefore $\bar{\Delta}(\pi)$ and $L(\pi)$) were *implicit*, with their existence established via categorification. Even the construction of simple cuspidals $L(\beta)$ was indirect. Here, we use *skew Specht modules* to render a more direct combinatorial description of semicuspidal and simple $R(\omega)$-modules.

Skew Specht modules

For each skew diagram $\boldsymbol{\tau}$ of content $\omega \in \mathbb{Z}_{\geqslant 0}I$, we can construct an associated skew Specht module, $\mathbf{S}^{\boldsymbol{\tau}}$. This is an $R(\omega)$-module, generalising Specht modules indexed by multipartitions; it has a presentation via generators and relations, and an explicit basis indexed by standard $\boldsymbol{\tau}$-tableaux. Specht modules are key objects in the representation theory of *cyclotomic* KLR algebras, Hecke algebras and symmetric groups.

Skew Specht modules

Setup: let t^{τ} denote the most dominant standard τ-tableau, i.e. the one formed by putting entries in order along the first row of the first component, then the second, etc. For example,



We can give nodes residues, as in the previous talk, up to a choice of 'starting residue' in each connected component. For example, if $e = 4$, we could have residues as above.

We define $\mathbf{i}^{\tau}$ to be the residue sequence obtained by reading these in order along rows. e.g. the above has
$\mathbf{i}^{\tau} = (1, 2, 2, 3, 0, 0, 1, 2, 3, 0, 3, 0, 1)$.

Skew Specht modules

Let $\boldsymbol{\tau}$ have content $\omega \in \mathbb{Z}_{\geqslant 0}I$. The skew Specht module $\mathbf{S}^{\boldsymbol{\tau}}$ is the graded $R(\omega)$-module generated by the vector $v^{\boldsymbol{\tau}}$ (which we set to be in degree zero) subject to the relations:

- $e(\mathbf{i})v^{\boldsymbol{\tau}} = \delta_{\mathbf{i},\mathbf{i}^{\boldsymbol{\tau}}} v^{\boldsymbol{\tau}}$ for all $\mathbf{i} \in I^{\omega}$;
- $x_r v^{\boldsymbol{\tau}} = 0$ for all r;
- $\psi_r v^{\boldsymbol{\tau}} = 0$ whenever r and $r+1$ are adjacent in $t^{\boldsymbol{\tau}}$;
- $g^u v^{\boldsymbol{\tau}} = 0$ for each $u \in \boldsymbol{\tau}$ with a node below it in $\boldsymbol{\tau}$ (i.e. for each 'Garnir node' u).

Theorem

$\mathbf{S}^{\boldsymbol{\tau}}$ *has a homogeneous basis* $\{v^t = \psi^t v^{\boldsymbol{\tau}} \mid t \in \mathrm{Std}(\boldsymbol{\tau})\}$.

Cuspidal modules as skew Specht modules

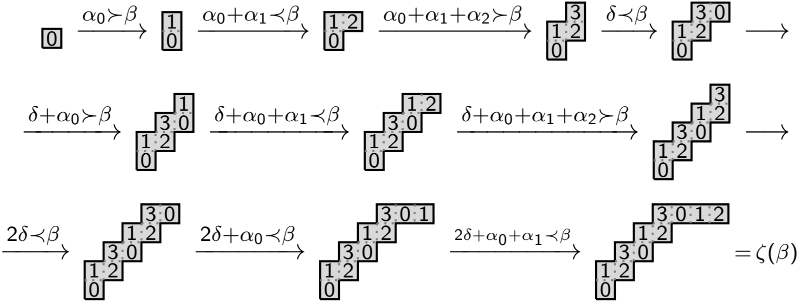
Theorem (ADMPSS, 2023 (Muth et al.))

For all $\beta \in \Phi_+^{\mathrm{re}}$, there exists an explicit ribbon $\zeta(\beta)$ of content β such that $\mathbf{S}^{\zeta(\beta)} \cong L(\beta)$.

To construct this ribbon, suppose that $\beta = \alpha_a + \alpha_{a+1} + \cdots + \alpha_b$. Start with an a-node, and iteratively add nodes of residues $a+1, a+2, \ldots, b$, where residues are taken modulo e. At each step the node is added either 1) above, if the current root is larger than β; or 2) to the right, if the current root is smaller than β.

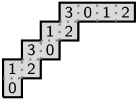
Example

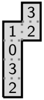
Take $e = 4$, and fix a certain convex preorder $\succcurlyeq$ on Φ_+ (see [MNSS]). Take $\beta = 2\delta + \alpha_0 + \alpha_1 + \alpha_2 = \alpha_0 + \alpha_1 + \cdots + \alpha_2 \in \Phi_+^{\mathrm{re}}$.

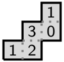


Example

This, and similar computations give

$$\zeta(2\delta+\alpha_0+\alpha_1+\alpha_2) =$$ 

$$\zeta(\delta+\alpha_2+\alpha_3) =$$ 

$$\zeta(\delta+\alpha_1) =$$ 

and the previous theorem tells us that $\mathbf{S}^{\zeta(\beta)} \cong L(\beta)$ in each case.

The new work I want to present today: we established an analogous result for the *imaginary* simple semicuspidal modules.

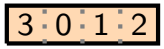
Imaginary simple semicuspidal modules

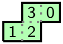
To each $(e-1)$-multipartition $\boldsymbol{\nu}$ of d, we construct a skew diagram $\zeta(\boldsymbol{\nu})$ of content $d\delta$, and show that $L(\boldsymbol{\nu}) \cong \mathrm{hd}\mathbf{S}^{\zeta(\boldsymbol{\nu})}$.

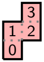
$\zeta(\boldsymbol{\nu})$ is constructed by 'dilating' the multipartition $\boldsymbol{\nu}$, replacing each node by a ribbon of length e. The ribbons are constructed exactly as in the case of real roots, but now we have e different choices of 'starting residue'. We label them $\zeta_0, \zeta_1, \dots, \zeta_{e-1}$ so that ζ_i has $i+1$ rows. Given $\boldsymbol{\nu} = (\nu^{(1)} \mid \nu^{(2)} \mid \cdots \mid \nu^{(e-1)})$, we dilate nodes in $\nu^{(i)}$ by the ribbon ζ_i. (We don't use ζ_0 at all.)

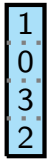
Example

Continuing with $e = 4$ and the same preorder as before, we construct distinct ribbons ζ_i of content δ:

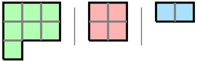
$\zeta_0 =$ 

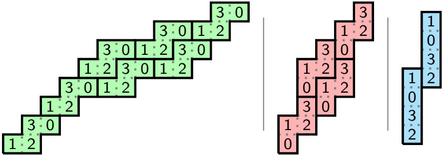
$\zeta_1 =$ 

$\zeta_2 =$ 

$\zeta_3 =$ 

and to any multipartition $\boldsymbol{\nu} = (\nu^{(1)} \mid \nu^{(2)} \mid \nu^{(3)})$, we associate the skew diagram $\zeta(\boldsymbol{\nu})$ by 'dilating' nodes in $\nu^{(i)}$ by the ribbon ζ_i. e.g. for $\boldsymbol{\nu} = ((3^2, 1) \mid (2^2) \mid (2))$, we have

$\zeta(\boldsymbol{\nu}) = \zeta\Big($  $\Big)$

$= \Big($  $\Big).$

Imaginary simple semicuspidal modules

Theorem (Muth–Nicewicz–S.–Sutton, 2024)

Let $\boldsymbol{\nu}$ be an $(e-1)$-multipartitions of d. Then:

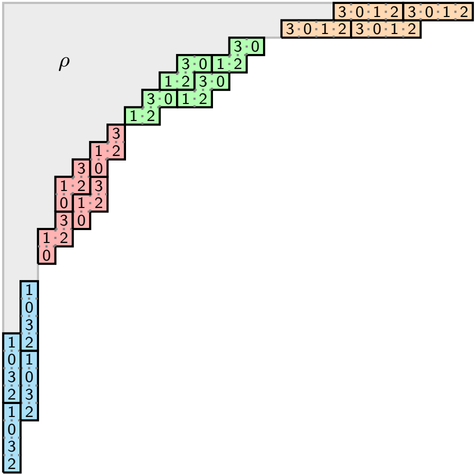
- *The skew Specht* $\mathbf{S}^{\zeta(\boldsymbol{\nu})}$ *is an indecomposable semicuspidal* $R(d\delta)$*-module, w/ simple semicuspidal head* $\mathrm{hd}(\mathbf{S}^{\zeta(\boldsymbol{\nu})})$.
- *Let* $L(\boldsymbol{\nu})$ *be the unique self-dual grading shift of* $\mathrm{hd}(\mathbf{S}^{\zeta(\boldsymbol{\nu})})$. *Then*

  $$\{L(\boldsymbol{\nu}) \mid \boldsymbol{\nu} \text{ an } (e-1)\text{-multipartition of } d\}$$

  is a complete and irredundant set of simple imaginary semicuspidal $R(d\delta)$*-modules, up to isomorphism & grading shift.*
- *For any* $(e-1)$*-multipartition* $\boldsymbol{\mu}$ *of* d*, we have* $[\mathbf{S}^{\zeta(\boldsymbol{\nu})} : L(\boldsymbol{\mu})] = d^{\mathrm{RoCK}}_{\boldsymbol{\nu},\boldsymbol{\mu}}$.

Connection with RoCK blocks

$d^{\mathsf{RoCK}}_{\boldsymbol{\nu},\boldsymbol{\mu}}$ is a decomposition number in a RoCK block, coming from adding the dilated shapes $\zeta(\boldsymbol{\nu})$ or $\zeta(\boldsymbol{\mu})$ to a large enough core ρ.



What about the other simple modules?

Now we can describe the simple cuspidal modules $L(\beta)$ for $\beta \in \Phi_{+}^{\mathrm{re}}$ as skew Specht modules, and the simple semicuspidal modules $L(\boldsymbol{\nu})$ as heads of skew Specht modules for $\boldsymbol{\nu}$ an $(e-1)$-multipartition. But what about general $L(\pi)$, for π a root partition of $\omega \in \mathbb{Z}_{\geqslant 0}I$?

Remember,

$$\begin{aligned} L(\pi) &= \mathrm{hd}\bar{\Delta}(\pi) \\ &= \mathrm{hd}(L(\beta_1)^{\circ K_{\beta_1}} \circ \cdots \circ L(\beta_u)^{\circ K_{\beta_u}} \circ L(\boldsymbol{\nu}) \circ L(\beta_{u+1})^{\circ K_{\beta_{u+1}}} \circ \cdots \circ L(\beta_t)^{\circ K_{\beta_t}}) \\ &= \mathrm{hd}(L(\beta_1)^{\circ K_{\beta_1}} \circ \cdots \circ L(\beta_u)^{\circ K_{\beta_u}} \circ \mathrm{hd}(\mathbf{S}^{\zeta(\boldsymbol{\nu})}) \circ L(\beta_{u+1})^{\circ K_{\beta_{u+1}}} \circ \cdots \circ L(\beta_t)^{\circ K_{\beta_t}}). \end{aligned}$$

This construction is still a little unsatisfying! Coming from a symmetric groups and cellular algebras perspective, I would really prefer a description of a simple module as being the head of a Specht-like module!

General simple modules

More generally, for each $\pi = (\boldsymbol{K}, \boldsymbol{\nu}) \in \Pi(\omega)$, we construct a skew diagram $\zeta(\pi)$ of content ω by concatenating semicuspidal skew diagrams with multiplicities determined by $\boldsymbol{K}$:

$$\zeta(\pi) = \left(\zeta(\beta_1)^{K_{\beta_1}} \mid \cdots \mid \zeta(\beta_u)^{K_{\beta_u}} \mid \zeta(\boldsymbol{\nu}) \mid \zeta(\beta_{u+1})^{K_{\beta_{u+1}}} \mid \cdots \mid \zeta(\beta_t)^{K_{\beta_t}}\right).$$

Theorem (Muth–Nicewicz–S.–Sutton, 2024)

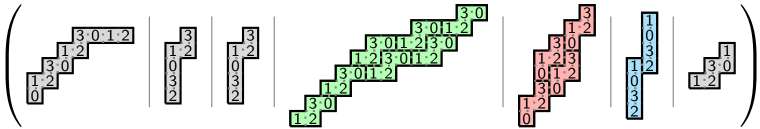
Let $\pi = (\boldsymbol{K}, \boldsymbol{\nu}) \in \Pi(\omega)$. Then:

- *The skew Specht $\mathbf{S}^{\zeta(\pi)}$ is indecomp. w/ simple head $\cong L(\pi)$; $\{\mathrm{hd}(\mathbf{S}^{\zeta(\pi)}) \mid \pi \in \Pi(\omega)\}$ gives a complete irredundant set of simple $R(\omega)$-modules up to grading shift.*
- *For all root partitions of the form $(\boldsymbol{K}, \boldsymbol{\mu}) \in \Pi(\omega)$, we have $[\mathbf{S}^{\zeta(\pi)} : L(\boldsymbol{K}, \boldsymbol{\mu})] = d^{\mathrm{RoCK}}_{\boldsymbol{\nu},\boldsymbol{\mu}}$.*
- *$\mathbf{S}^{\zeta(\pi)} \twoheadrightarrow \bar{\Delta}(\pi)$; $\mathbf{S}^{\zeta(\pi)}$ has a filtration by proper standard modules of the form $\bar{\Delta}(\boldsymbol{K}, \boldsymbol{\mu})$, & $(\mathbf{S}^{\zeta(\pi)} : \bar{\Delta}(\boldsymbol{K}, \boldsymbol{\mu})) = d^{\mathrm{RoCK}}_{\boldsymbol{\nu},\boldsymbol{\mu}}$.*

Example

Take $\pi \in \Pi(19\alpha_0 + 20\alpha_1 + 21\alpha_2 + 20\alpha_3)$ defined as

$$\pi = \left(\left(2\delta + \alpha_0 + \alpha_1 + \alpha_2 \mid (\delta + \alpha_2 + \alpha_3)^2 \mid \delta^{13} \mid \delta + \alpha_1\right), \boldsymbol{\nu}\right).$$ Then

$\zeta(\pi) =$ 

Our theorem says that $\mathbf{S}^{\zeta(\pi)}$ is an indecomposable $R(19\alpha_0 + 20\alpha_1 + 21\alpha_2 + 20\alpha_3)$-module with simple head $L(\pi)$.