GENERAL RELATIVITY – FINAL EXAM

Exercise 1. *Consider a particle with mass m moving under the influence of gravity alone, in a spatially flat Friedmann universe:*

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \ . \tag{1}$$

Let τ be the particle's proper time, and $\varepsilon \equiv -mu_t = m\frac{dt}{d\tau}$ its energy.

1. *Express ε as a function of the scale factor a throughout the motion. As initial data, assume $\varepsilon = \varepsilon_0$ at some initial value $a = a_0$. Hint: there are two possible methods: either use the geodesic equation, or momentum conservation.*

2. *What does the dependence $\varepsilon(a)$ become in the limits $\varepsilon \approx m$ and $\varepsilon \gg m$?*

Exercise 2. *Now, suppose that the stress-energy tensor in the universe (1) takes the form:*

$$T^t_t = -\rho(t) \ ; \quad T^j_i = p(t)\delta^j_i \ ; \quad T^t_i = T^i_t = 0 \ , \tag{2}$$

where ρ is energy density, and p is pressure. We consider two cases:

- *Radiation: $p = \rho/3$, so that $T^\mu_\mu = 0$ (c.f. Exercise 3 in Homework 4).*
- *Vacuum energy: $p = -\rho$, so that $T^\nu_\mu = -\rho\,\delta^\nu_\mu$.*

For each of these two cases:

1. *Using the covariant conservation law $\nabla_\nu T^\nu_\mu = 0$, find the dependence of ρ on the scale factor a. As initial data, assume $\rho = \rho_0$ at some initial value $a = a_0$.*

2. *Using the Einstein equations, find $a(t)$.*

One of your answers is directly related to one of the answers in Exercise 1. Explain!

Exercise 3. *Now, consider a spatially spherical Friedmann universe:*

$$ds^2 = -dt^2 + a^2(t)\big(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)\big) \ . \tag{3}$$

The stress-energy tensor is again given by (2), where the spatial indices i, j now run over the values (χ, θ, ϕ).

1. *Compute the non-vanishing components of the Ricci tensor:*

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\rho} \; . \tag{4}$$

2. *For what values of ρ and p do the Einstein equations admit a static solution $a(t) =$ const? Recall from Exercise 2 that negative p is a possibility!*

Exercise 4. *Consider the gravitational field of a Schwarzschild black hole:*

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \; . \tag{5}$$

A particle is orbiting the black hole (under the influence of gravity alone) at some radius $r =$ const, in the equatorial plane $\theta = \pi/2$.

1. *Find the time interval Δt for a complete orbit.* <u>*Hint: to find the trajectory $\phi(t)$, consider the geodesic equation $\Gamma^\mu_{\nu\rho} dx^\nu dx^\rho = 0$.*</u>

2. *Find the proper time interval $\Delta\tau$ for a complete orbit.*

3. *What is the smallest radius r_{min} at which such circular orbits are possible?*